L11 Problem Set 1

Ramana Kumar (rk436)

October 25, 2010

1 $\mathbb{M}_n(S)$ is $(n-1)$-stable if S is 0-stable

That S is 0-stable means if $a \in S$ then $a^{(0)} = a^{(1)}$. Expanding the sums of powers, we have $\bar{1} = a^0 = a^{(0)} = a^{(1)} = a^0 \oplus a^1 = \bar{1} \oplus a$. Thus, $\bar{1}$ is an annihilator for $\oplus$.

Let $\mathbf{A} \in \mathbb{M}_n(S)$. This matrix can be seen to encode a path problem, by taking the graph with vertexes $\{1, \ldots, n\}$, edges $\{(i,j) \mid \mathbf{A}(i,j) \neq \bar{0}\}$, and weight function $w(i,j) = \mathbf{A}(i,j)$. We also assume that S is a semiring. Therefore the theorem on slide 42 is applicable, and we can write entries of the kth power of $\mathbf{A}$ as

$$\mathbf{A}^k(i,j) = \bigoplus_{p \in P^k(i,j)} w(p)$$

where $P^k(i,j)$ is the set of paths $(i, q_1, q_2, \ldots, q_{k-1}, j)$ from i to j with exactly k arcs, and

$$\begin{aligned} w(i, q_1, \ldots, q_{k-1}, j) &= \mathbf{A}(i, q_1) \otimes \cdots \otimes \mathbf{A}(q_{k-1}, j), \text{ with} \\ w(i,i) &= \bar{1} \end{aligned}$$

The theorem on slide 42 also gives the entries of $\mathbf{A}^{(k)}$:

$$\mathbf{A}^{(k)}(i,j) = \bigoplus_{p \in P^{(k)}(i,j)} w(p)$$

where $P^{(k)}(i,j)$ is the set of paths with at most k arcs, in symbols $P^{(k)}(i,j) = \bigcup_{0 \leq h \leq k} P^h(i,j)$.

Now consider the entries of $\mathbf{A}^{(n)}$:

$$\begin{aligned} \mathbf{A}^{(n)}(i,j) &= \mathbf{A}^{(n-1)}(i,j) \oplus \mathbf{A}^n(i,j) \\ &= \mathbf{A}^{(n-1)}(i,j) \oplus \bigoplus_{p \in P^n(i,j)} w(p) \end{aligned}$$

Every path $p = (q_0, q_1, \ldots, q_{n-1}, q_n) \in P^n(i,j)$, where $q_0 = i$ and $q_n = j$, contains $n+1$ vertexes, each drawn from the set of n vertexes. It follows (from the pigeonhole principle) that there are distinct indexes $a < b$ such that $q_a = q_b$. Let $c = (q_a, \ldots, q_b)$ be the cyclic subpath, and let $q_{0\ldots a}$ denote the prefix $(q_0, \ldots, q_a)$ and $q_{b\ldots n}$ the suffix $(q_b, \ldots, q_n)$. Isolating c, we can write[1]

$$\mathbf{A}^{(n)}(i,j) = \mathbf{A}^{(n-1)}(i,j) \oplus \bigoplus_{(q_0, \ldots, q_n) \in P^n(i,j)} w(q_{0\ldots a}) \otimes w(c) \otimes w(q_{b\ldots n}) \qquad (*)$$

[1] The dependence of a, b, and c, on the path p is not apparent in the notation.

The path $\lfloor p \rfloor = q_{0\ldots a} q_{b\ldots n}$, with c removed, has $n - (b-a) \leq n-1$ arcs, so is a member of $P^{(n-1)}(i,j)$. It follows that $w(\lfloor p \rfloor)$ is a term in the summation for $\mathbf{A}^{(n-1)}(i,j)$. This term can be simplified as

$$
\begin{aligned}
w(\lfloor p \rfloor) &= w(q_{0\ldots a}) \otimes w(q_a, q_b) \otimes w(q_{b\ldots n}) \\
&= w(q_{0\ldots a}) \otimes \bar{1} \otimes w(q_{b\ldots n}) \\
&= w(q_{0\ldots a}) \otimes w(q_{b\ldots n})
\end{aligned}
$$

There may be many paths in $P^n(i,j)$ containing subpaths $q_{0\ldots a}$ and $q_{b\ldots n}$, each with a different $(b-a)$-arc cycle $(q_a, \ldots, q_b)$ connecting them. There are finitely many such cycles, so let $c_0, c_1, \ldots, c_m$ enumerate them and let $p_0, p_1, \ldots, p_m$ denote the n-arc paths containing each cycle. We can rearrange and simplify the whole sum according to the following schema.

$$
\begin{aligned}
\mathbf{A}^{(n)}(i,j) &= \ldots \oplus w(\lfloor p \rfloor) \oplus \left(\bigoplus_{0 \leq k \leq m} w(p_k) \right) \oplus \ldots \\
&= \ldots \oplus w(q_{0\ldots a}) \otimes w(q_{b\ldots n}) \oplus \left(\bigoplus_{0 \leq k \leq m} w(q_{0\ldots a}) \otimes w(c_k) \otimes w(q_{b\ldots n}) \right) \oplus \ldots \\
&= \ldots \oplus w(q_{0\ldots a}) \otimes w(q_{b\ldots n}) \oplus w(q_{0\ldots a}) \otimes \left(\bigoplus_{0 \leq k \leq m} w(c_k) \right) \otimes w(q_{b\ldots n}) \oplus \ldots && \text{(distributivity)} \\
&= \ldots \oplus w(q_{0\ldots a}) \otimes \left[w(q_{b\ldots n}) \oplus \left(\bigoplus_{0 \leq k \leq n} w(c_k) \right) \otimes w(q_{b\ldots n}) \right] \oplus \ldots && \text{(distributivity)} \\
&= \ldots \oplus w(q_{0\ldots a}) \otimes \left[\left[\bar{1} \oplus \left(\bigoplus_{0 \leq k \leq n} w(c_k) \right) \right] \otimes w(q_{b\ldots n}) \right] \oplus \ldots && \text{(distributivity)} \\
&= \ldots \oplus w(q_{0\ldots a}) \otimes [\bar{1} \otimes w(q_{b\ldots n})] \oplus \ldots && (\bar{1} \text{ is an annihilator}) \\
&= \ldots \oplus w(q_{0\ldots a}) \otimes w(q_{b\ldots n}) \oplus \ldots \\
&= \ldots \oplus w(\lfloor p \rfloor) \oplus \ldots
\end{aligned}
$$

In words: the weights of paths with common prefix and suffix but different cycles are all absorbed by the weight of the path with the cycle removed.

We have established that all paths in $P^n(i,j)$ contain a cycle, but in general they may contain many cycles. The schema above shows how, after isolating a particular cycle, the weights of all n-arc paths with the same prefix and suffix are absorbed in the summation for $\mathbf{A}^{(n)}(i,j)$. In fact, the choice of cycle does not matter: by arbitrarily choosing one cycle per path in $P^n(i,j)$, then grouping paths with the same prefixes and suffixes, we can see how none of those paths affects the sum of weights. Thus in $(*)$ the terms involving $w(c)$ are absorbed, leaving $\mathbf{A}^{(n)}(i,j) = \mathbf{A}^{(n-1)}(i,j)$. Since $\mathbf{A}$ is arbitrary, we conclude $\mathbb{M}_n(S)$ is $(n-1)$-stable.

2 $\mathbb{M}_n(S)$ is a semiring

We assume $(S, \oplus, \otimes, \bar{0}, \bar{1})$ is a semiring.

2.1 $(\mathbb{M}_n(S), \oplus, \mathbf{J})$ is a commutative monoid

2.1.1 $\mathbb{M}_n(S)$ is non-empty

We know $\bar{0} \in S$, and $\mathbf{J} \in \mathbb{M}_n(S)$ follows.

2.1.2 Associativity

$$\begin{aligned}(\mathbf{A} \oplus (\mathbf{B} \oplus \mathbf{C}))(i,j) = \mathbf{A}(i,j) \oplus (\mathbf{B} \oplus \mathbf{C})(i,j) = \\ \mathbf{A}(i,j) \oplus (\mathbf{B}(i,j) \oplus \mathbf{C}(i,j)) = (\mathbf{A}(i,j) \oplus \mathbf{B}(i,j)) \oplus \mathbf{C}(i,j) \\ = (\mathbf{A} \oplus \mathbf{B})(i,j) \oplus \mathbf{C}(i,j) = ((\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C})(i,j)\end{aligned}$$

2.1.3 Identity

$$\begin{aligned}(\mathbf{A} \oplus \mathbf{J})(i,j) = \mathbf{A}(i,j) \oplus \mathbf{J}(i,j) = \\ \mathbf{A}(i,j) \oplus \bar{0} = \mathbf{A}(i,j) = \bar{0} \oplus \mathbf{A}(i,j) \\ = \mathbf{J}(i,j) \oplus \mathbf{A}(i,j) = (\mathbf{J} \oplus \mathbf{A})(i,j)\end{aligned}$$

2.1.4 Commutativity

$$(\mathbf{A} \oplus \mathbf{B})(i,j) = \mathbf{A}(i,j) \oplus \mathbf{B}(i,j) = \mathbf{B}(i,j) \oplus \mathbf{A}(i,j) = (\mathbf{B} \oplus \mathbf{A})(i,j)$$

2.2 $(\mathbb{M}_n(S), \otimes, \mathbf{I})$ is a monoid

2.2.1 $\mathbb{M}_n(S)$ is non-empty

As above (2.1.1).

2.2.2 Associativity

$$
\begin{aligned}
(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))(i,j) &= \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes (\mathbf{B} \otimes \mathbf{C})(q,j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \left(\bigoplus_{1 \le r \le n} \mathbf{B}(q,r) \otimes \mathbf{C}(r,j) \right) \\
&= \bigoplus_{1 \le q \le n} \bigoplus_{1 \le r \le n} \mathbf{A}(i,q) \otimes \mathbf{B}(q,r) \otimes \mathbf{C}(r,j) && \text{(distributivity, and } \otimes_S \text{ is associative)} \\
&= \bigoplus_{1 \le r \le n} \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \mathbf{B}(q,r) \otimes \mathbf{C}(r,j) && (\oplus_S \text{ is associative and commutative)} \\
&= \bigoplus_{1 \le r \le n} \left(\bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \mathbf{B}(q,r) \right) \otimes \mathbf{C}(r,j) && \text{(distributivity)} \\
&= \bigoplus_{1 \le r \le n} (\mathbf{A} \otimes \mathbf{B})(i,r) \otimes \mathbf{C}(r,j) \\
&= ((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C})(i,j)
\end{aligned}
$$

2.2.3 Identity

$$
\begin{gathered}
(\mathbf{A} \otimes \mathbf{I})(i,j) = \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \mathbf{I}(q,j) = \\
\mathbf{A}(i,1) \otimes \bar{0} \oplus \mathbf{A}(i,2) \otimes \bar{0} \oplus \cdots \oplus \mathbf{A}(i,j) \otimes \bar{1} \oplus \cdots \oplus \mathbf{A}(i,n-1) \otimes \bar{0} \oplus \mathbf{A}(i,n) \otimes \bar{0} = \\
\bar{0} \oplus \cdots \oplus \mathbf{A}(i,j) \oplus \cdots \oplus \bar{0} = \\
\mathbf{A}(i,j) \\
= \bar{0} \oplus \cdots \oplus \mathbf{A}(i,j) \oplus \cdots \oplus \bar{0} \\
= \bar{0} \otimes \mathbf{A}(i,1) \oplus \bar{0} \otimes \mathbf{A}(i,2) \oplus \cdots \oplus \bar{1} \otimes \mathbf{A}(i,j) \oplus \cdots \oplus \bar{0} \otimes \mathbf{A}(i,n-1) \oplus \bar{0} \otimes \mathbf{A}(i,n) \\
= \bigoplus_{1 \le q \le n} \mathbf{I}(i,q) \otimes \mathbf{A}(q,j) = (\mathbf{I} \otimes \mathbf{A})(i,j)
\end{gathered}
$$

2.3 J is an annihilator for $\otimes$

$$
\begin{gathered}
(\mathbf{J} \otimes \mathbf{A})(i,j) = \bigoplus_{1 \le q \le n} \mathbf{J}(i,q) \otimes \mathbf{A}(q,j) = \bigoplus_{1 \le q \le n} \bar{0} \otimes \mathbf{A}(q,j) = \bigoplus_{1 \le q \le n} \bar{0} = \bar{0} = \\
\mathbf{J}(i,j) \\
= \bar{0} = \bigoplus_{1 \le q \le n} \bar{0} = \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \bar{0} = \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \mathbf{J}(q,j) = (\mathbf{A} \otimes \mathbf{J})(i,j)
\end{gathered}
$$

2.4 Distributivity

Slide 39 gives left distributivity: $\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} \oplus \mathbf{A} \otimes \mathbf{C}$. Right distributivity is below.

$$
\begin{aligned}
((\mathbf{A} \oplus \mathbf{B}) \otimes \mathbf{C})(i,j) &= \bigoplus_{1 \le q \le n} (\mathbf{A} \oplus \mathbf{B})(i,q) \otimes \mathbf{C}(q,j) \\
&= \bigoplus_{1 \le q \le n} (\mathbf{A}(i,q) \oplus \mathbf{B}(i,q)) \otimes \mathbf{C}(q,j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \mathbf{C}(q,j) \oplus \mathbf{B}(i,q) \otimes \mathbf{C}(q,j) \\
&= \left(\bigoplus_{1 \le q \le n} \mathbf{A}(i,q) \otimes \mathbf{C}(q,j) \right) \oplus \left(\bigoplus_{1 \le q \le n} \mathbf{B}(i,q) \otimes \mathbf{C}(q,j) \right) \\
&= (\mathbf{A} \otimes \mathbf{C})(i,j) \oplus (\mathbf{B} \otimes \mathbf{C})(i,j) \\
&= (\mathbf{A} \otimes \mathbf{C} \oplus \mathbf{B} \otimes \mathbf{C})(i,j)
\end{aligned}
$$

3 $(S, \oplus_S)\vec{\times}(T, \oplus_T)$ is associative

Assume $(S, \oplus_S)$ is a commutative idempotent semigroup, and $(T, \oplus_T)$ is a monoid. Associativity of $\vec{\oplus}$ amounts to

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

There are sixteen cases:

- $s_1 = s_1 \oplus_S s_2 = s_2$ and $s_2 = s_2 \oplus_S s_3 = s_3$
  Let $s = s_1 = s_2 = s_3$. Clearly $s = s \oplus_S s = s$. Then,

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s,t_3) = (s \oplus_S s, t_1 \oplus_T t_2)\vec{\oplus}(s,t_3) =$$
$$(s, t_1 \oplus_T t_2)\vec{\oplus}(s,t_3) = (s \oplus_S s, (t_1 \oplus_T t_2) \oplus_T t_3) = (s \oplus_S s, t_1 \oplus_T (t_2 \oplus_T t_3)) = (s,t_1)\vec{\oplus}(s, t_2 \oplus_T t_3)$$
$$= (s,t_1)\vec{\oplus}(s \oplus_S s, t_2 \oplus_T t_3) = (s,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 = s_2$ and $s_2 = s_2 \oplus_S s_3 \neq s_3$
  Let $s = s_1 = s_2$. Clearly $s = s \oplus_S s = s$, and $s = s \oplus_S s_3 \neq s_3$. Then,

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s_3,t_3) = (s \oplus_S s, t_1 \oplus_T t_2)\vec{\oplus}(s_3,t_3) =$$
$$(s, t_1 \oplus_T t_2)\vec{\oplus}(s_3,t_3) = (s \oplus_S s_3, t_1 \oplus_T t_2) = (s, t_1 \oplus_T t_2) = (s \oplus_S s, t_1 \oplus_T t_2) = (s,t_1)\vec{\oplus}(s,t_2)$$
$$= (s,t_1)\vec{\oplus}(s \oplus s_3, t_2) = (s,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s_3,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 = s_2$ and $s_2 \neq s_2 \oplus_S s_3 = s_3$
  Let $s = s_1 = s_2$. Clearly $s = s \oplus_S s = s$, and $s \neq s \oplus_S s_3 = s_3$. Then,

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s_3,t_3) = (s \oplus_S s, t_1 \oplus_T t_2)\vec{\oplus}(s_3,t_3) =$$
$$(s, t_1 \oplus_T t_2)\vec{\oplus}(s_3,t_3) = (s \oplus_S s_3, t_3) = (s,t_1)\vec{\oplus}(s_3,t_3)$$
$$= (s,t_1)\vec{\oplus}(s \oplus s_3, t_3) = (s,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s_3,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 = s_2$ and $s_2 \neq s_2 \oplus_S s_3 \neq s_3$[2]
  Let $s = s_1 = s_2$. Clearly $s = s \oplus_S s = s$, and $s \neq s \oplus_S s_3 \neq s_3$. We also infer $s \oplus_S (s \oplus_S s_3) = (s \oplus_S s) \oplus_S s_3 = s \oplus_S s_3$, and therefore $s \neq s \oplus_S (s \oplus_S s_3) = s \oplus_S s_3$. Then,

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s_3,t_3) = (s \oplus_S s, t_1 \oplus_T t_2)\vec{\oplus}(s_3,t_3) =$$
$$(s, t_1 \oplus_T t_2)\vec{\oplus}(s_3,t_3) = (s \oplus_S s_3, \bar{0}_T) = (s \oplus_S (s \oplus_S s_3), \bar{0}_T)$$
$$= (s,t_1)\vec{\oplus}(s \oplus_S s_3, \bar{0}_T) = (s,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s_3,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 \neq s_2$ and $s_2 = s_2 \oplus_S s_3 = s_3$
  Let $s = s_2 = s_3$. Clearly $s = s \oplus_S s = s$, and $s_1 = s_1 \oplus_S s \neq s$. Then,

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s_1,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s,t_3) = (s_1 \oplus_S s, t_1)\vec{\oplus}(s,t_3) =$$
$$(s_1,t_1)\vec{\oplus}(s,t_3) = (s_1 \oplus_S s, t_1) = (s_1,t_1)\vec{\oplus}(s, t_2 \oplus_T t_3)$$
$$= (s_1,t_1)\vec{\oplus}(s \oplus_S s, t_2 \oplus_T t_3) = (s_1,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 \neq s_2$ and $s_2 = s_2 \oplus_S s_3 \neq s_3$
  We infer $s_1 \oplus_S s_3 = (s_1 \oplus_S s_2) \oplus_S s_3 = s_1 \oplus_S (s_2 \oplus_S s_3) = s_1 \oplus_S s_2 = s_1$. We also infer that $s_1 \neq s_3$, since otherwise we would have $s_3 \oplus_S s_2 = s_1 \oplus_S s_2 \neq s_2 = s_2 \oplus_S s_3$, but $\oplus_S$ is commutative. Thus $s_1 = s_1 \oplus_S s_3 \neq s_3$. Then,

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_1)\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}(s_3,t_3) =$$
$$(s_1 \oplus_S s_3, t_1) = (s_1,t_1) = (s_1 \oplus_S s_2, t_1)$$
$$= (s_1,t_1)\vec{\oplus}(s_2,t_2) = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_2) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 \neq s_2$ and $s_2 \neq s_2 \oplus_S s_3 = s_3$

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_1)\vec{\oplus}(s_3,t_3) =$$
$$(s_1,t_1)\vec{\oplus}(s_3,t_3)$$
$$= (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_3) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

- $s_1 = s_1 \oplus_S s_2 \neq s_2$ and $s_2 \neq s_2 \oplus_S s_3 \neq s_3$
  We infer $s_1 \oplus_S s_3 \neq s_3$, since otherwise, using commutativity and associativity of $\oplus_S$, we would have $s_2 \oplus_S s_3 = s_2 \oplus_S s_1 \oplus_S s_3 = s_1 \oplus_S s_3 = s_3$. We also infer $s_1 \oplus_S s_3 = s_1 \oplus_S (s_2 \oplus_S s_3)$. There are two major cases, each split into two minor cases:

  - $s_1 = s_1 \oplus_S s_3 \neq s_3$ and $s_1 = s_1 \oplus_S (s_2 \oplus_S s_3) =_? s_2 \oplus_S s_3$
    If the $=_?$ is $\neq$, remove the $[\oplus_T \bar{0}_T]_?$, otherwise keep it. In both cases the result is the same.

$$\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_1)\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}(s_3,t_3) =$$
$$(s_1 \oplus_S s_3, t_1) = (s_1 \oplus_S (s_2 \oplus_S s_3), t_1[\oplus_T \bar{0}_T]_?)$$
$$= (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, \bar{0}_T) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]$$

[2] "$a \neq b \neq c$" is shorthand for "$a \neq b$ and $b \neq c$" and does not imply any relationship between a and c.

– $s_1 \neq s_1 \oplus_S s_3$ and $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3) =_? s_2 \oplus_S s_3$
In both cases the result is the same.

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_1)\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}(s_3,t_3) = \\ (s_1 \oplus_S s_3, \bar{0}_T) = (s_1 \oplus_S (s_2 \oplus_S s_3), \bar{0}_T) \\ = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, \bar{0}_T) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 = s_2$ and $s_2 = s_2 \oplus_S s_3 = s_3$
Let $s = s_2 = s_3$. Clearly $s = s \oplus_S s = s$, and $s_1 \neq s_1 \oplus_S s = s$. Then,

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s_1,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s,t_3) = (s_1 \oplus_S s, t_2)\vec{\oplus}(s,t_3) = \\ (s,t_2)\vec{\oplus}(s,t_3) = (s \oplus_S s, t_2 \oplus_T t_3) = (s, t_2 \oplus_T t_3) = (s_1 \oplus_S s, t_2 \oplus_T t_3) = (s_1,t_1)\vec{\oplus}(s, t_2 \oplus_T t_3) \\ = (s_1,t_1)\vec{\oplus}(s \oplus_S s, t_2 \oplus_T t_3) = (s_1,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 = s_2$ and $s_2 = s_2 \oplus_S s_3 \neq s_3$

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_2)\vec{\oplus}(s_3,t_3) = (s_2,t_2)\vec{\oplus}(s_3,t_3) = \\ (s_2 \oplus_S s_3, t_2) = (s_2, t_2) = (s_1 \oplus_S s_2, t_2) \\ = (s_1,t_1)\vec{\oplus}(s_2,t_2) = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_2) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 = s_2$ and $s_2 \neq s_2 \oplus_S s_3 = s_3$
We infer, using associativity, that $s_1 \oplus_S s_3 = s_1 \oplus_S s_2 \oplus_S s_3 = s_2 \oplus_S s_3 = s_3$. We also infer $s_1 \neq s_3$, since otherwise $s_2 = s_1 \oplus_S s_2 = s_3 \oplus_S s_2 = s_2 \oplus_S s_3 = s_3$. Thus $s_1 \neq s_1 \oplus_S s_3 = s_3$. Then,

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_2)\vec{\oplus}(s_3,t_3) = (s_2,t_2)\vec{\oplus}(s_3,t_3) = \\ (s_2 \oplus_S s_3, t_3) = (s_3, t_3) = (s_1 \oplus_S s_3, t_3) \\ = (s_1,t_1)\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_3) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 = s_2$ and $s_2 \neq s_2 \oplus_S s_3 \neq s_3$
We infer $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3)$, since otherwise, using the commutativity, associativity, and idempotence of $\oplus_S$, we would have $s_1 \oplus_S s_2 = s_1 \oplus_S s_2 \oplus_S s_3 \oplus_S s_2 = s_1 \oplus_S s_2 \oplus_S s_3 = s_1$. We also observe $s_1 \oplus_S (s_2 \oplus_S s_3) = (s_1 \oplus_S s_2) \oplus_S s_3 = s_2 \oplus_S s_3$. Thus, $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3) = s_2 \oplus_S s_3$. Then,

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = (s_1 \oplus_S s_2, t_2)\vec{\oplus}(s_3,t_3) = (s_2,t_2)\vec{\oplus}(s_3,t_3) = \\ (s_2 \oplus_S s_3, \bar{0}_T) = (s_1 \oplus_S (s_2 \oplus_S s_3), \bar{0}_T) \\ = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, \bar{0}_T) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 \neq s_2$ and $s_2 = s_2 \oplus_S s_3 = s_3$
Let $s = s_2 = s_3$. Clearly $s = s \oplus_S s = s$. We infer $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3)$ as above, thus $s_1 \neq s_1 \oplus_S s \neq s$. We also infer, using associativity, that $s_1 \oplus_S s = s_1 \oplus_S s \oplus_S s \neq s$. Then,

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) = \left[(s_1,t_1)\vec{\oplus}(s,t_2)\right]\vec{\oplus}(s,t_3) = (s_1 \oplus_S s, \bar{0}_T)\vec{\oplus}(s,t_3) = \\ (s_1 \oplus_S s \oplus_S s, \bar{0}_T) = (s_1 \oplus_S s, \bar{0}_T) = (s_1,t_1)\vec{\oplus}(s, t_2 \oplus_T t_3) \\ = (s_1,t_1)\vec{\oplus}(s \oplus_S s, t_2 \oplus_T t_3) = (s_1,t_1)\vec{\oplus}\left[(s,t_2)\vec{\oplus}(s,t_3)\right] = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 \neq s_2$ and $s_2 = s_2 \oplus_S s_3 \neq s_3$
  Using associativity, we observe $s_1 \oplus_S s_2 = s_1 \oplus_S s_2 \oplus_S s_3$. We infer $s_1 \oplus_S s_2 \neq s_3$, since otherwise, using commutativity and idempotence too, we would have $s_2 = s_2 \oplus_S s_3 = s_2 \oplus_S s_1 \oplus_S s_2 = s_1 \oplus_S s_2 \neq s_2$. Thus, $s_1 \oplus_S s_2 = (s_1 \oplus_S s_2) \oplus_S s_3 \neq s_3$. Then,

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) &= (s_1 \oplus_S s_2, \bar{0}_T)\vec{\oplus}(s_3,t_3) = \\ &((s_1 \oplus_S s_2) \oplus_S s_3, \bar{0}_T) = (s_1 \oplus_S s_2, \bar{0}_T) \\ &= (s_1,t_1)\vec{\oplus}(s_2,t_2) = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_2) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 \neq s_2$ and $s_2 \neq s_2 \oplus_S s_3 = s_3$
  We infer $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3)$ as above (three cases up), so $s_1 \neq s_1 \oplus_S s_3$. We then observe, using associativity, that $(s_1 \oplus_S s_2) \oplus_S s_3 = s_1 \oplus_S s_2 \oplus_S s_3 = s_1 \oplus_S s_3$. There are two major cases, each split into two minor cases:

  - $s_1 \neq s_1 \oplus_S s_3 = s_3$ and $s_1 \oplus_S s_2 =_? (s_1 \oplus_S s_2) \oplus_S s_3 = s_3$
    If the $=_?$ is $\neq$, remove the $[\bar{0}_T \oplus_T]_?$, otherwise keep it. In both cases the result is the same.

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) &= (s_1 \oplus_S s_2, \bar{0}_T)\vec{\oplus}(s_3,t_3) = \\ &((s_1 \oplus_S s_2) \oplus_S s_3, [\bar{0}_T \oplus_T]_? t_3) = (s_1 \oplus_S s_3, t_3) \\ &= (s_1,t_1)\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_3) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

  - $s_1 \neq s_1 \oplus_S s_3 \neq s_3$ and $s_1 \oplus_S s_2 =_? (s_1 \oplus_S s_2) \oplus_S s_3 \neq s_3$
    In both cases the result is the same.

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) &= (s_1 \oplus_S s_2, \bar{0}_T)\vec{\oplus}(s_3,t_3) = \\ &((s_1 \oplus_S s_2) \oplus_S s_3, \bar{0}_T) = (s_1 \oplus_S s_3, \bar{0}_T) \\ &= (s_1,t_1)\vec{\oplus}(s_3,t_3) = (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, t_3) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

- $s_1 \neq s_1 \oplus_S s_2 \neq s_2$ and $s_2 \neq s_2 \oplus_S s_3 \neq s_3$
  We infer $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3)$ as above. We also infer $(s_1 \oplus_S s_2) \oplus_S s_3 \neq s_3$, since otherwise $s_2 \oplus_S s_3 = s_2 \oplus_S s_1 \oplus_S s_2 \oplus_S s_3 = s_1 \oplus_S s_2 \oplus_S s_3 = s_3$. Therefore, we have $s_1 \neq s_1 \oplus_S (s_2 \oplus_S s_3) =_? s_2 \oplus_S s_3$ and $s_1 \oplus_S s_2 =_? (s_1 \oplus_S s_2) \oplus_S s_3 \neq s_3$. The $=_?$ means either $=$ or $\neq$, but, for our uses below, the result is $\bar{0}_T$ in the second component either way. Thus,

$$\begin{aligned}\left[(s_1,t_1)\vec{\oplus}(s_2,t_2)\right]\vec{\oplus}(s_3,t_3) &= (s_1 \oplus_S s_2, \bar{0}_T)\vec{\oplus}(s_3,t_3) = \\ &((s_1 \oplus_S s_2) \oplus_S s_3, \bar{0}_T) = (s_1 \oplus_S (s_2 \oplus_S s_3), \bar{0}_T) \\ &= (s_1,t_1)\vec{\oplus}(s_2 \oplus_S s_3, \bar{0}_T) = (s_1,t_1)\vec{\oplus}\left[(s_2,t_2)\vec{\oplus}(s_3,t_3)\right]\end{aligned}$$

4 Natural orders for the lexicographic product

Suppose S is a commutative idempotent semigroup and T is a monoid. The natural orders of $(S, \oplus_S)\vec{\times}(T, \oplus_T)$ are

$$\begin{aligned}
(s_1, t_1) \leq^L_{\vec{\oplus}} (s_2, t_2) \quad &\equiv \quad (s_1, t_1) = (s_1, t_1)\vec{\oplus}(s_2, t_2) \\
&\iff s_1 = s_1 \oplus_S s_2 \wedge \\
&\qquad ((s_1 \oplus_S s_2 = s_2 \wedge t_1 = t_1 \oplus_T t_2) \vee \\
&\qquad\; s_1 \oplus_S s_2 \neq s_2) \\
&\iff s_1 = s_1 \oplus_S s_2 \wedge (s_1 = s_2 \Rightarrow t_1 = t_1 \oplus_T t_2) \\
&\iff s_1 \leq^L_{\oplus_S} s_2 \wedge (s_1 = s_2 \Rightarrow t_1 \leq^L_{\oplus_T} t_2), \text{ and} \\
(s_1, t_1) \leq^R_{\vec{\oplus}} (s_2, t_2) \quad &\equiv \quad (s_2, t_2) = (s_1, t_1)\vec{\oplus}(s_2, t_2) \\
&\iff s_1 \oplus_S s_2 = s_2 \wedge \\
&\qquad ((s_1 = s_1 \oplus_S s_2 \wedge t_1 \oplus_T t_2 = t_2) \vee \\
&\qquad\; s_1 \neq s_1 \oplus_S s_2) \\
&\iff s_1 \oplus_S s_2 = s_2 \wedge (s_1 = s_2 \Rightarrow t_1 \oplus_T t_2 = t_2) \\
&\iff s_1 \leq^R_{\oplus_S} s_2 \wedge (s_1 = s_2 \Rightarrow t_1 \leq^R_{\oplus_T} t_2)
\end{aligned}$$

The order $\leq^L_{\vec{\oplus}}$ is the usual lexicographic order, built from the natural orders $\leq^L_{\oplus_S}$ and $\leq^L_{\oplus_T}$. In words: Compare the first components. If the first components are equal, then compare the second components. In both comparisons, the first argument must be at least as good as the second.

The order $\leq^R_{\vec{\oplus}}$ is the reverse lexicographic order, requiring instead that the second argument be at least as good as the first. It can be seen as the lexicographic order built from $\leq^R_{\oplus_S}$ and $\leq^R_{\oplus_T}$. All interesting properties of $\leq^R_{\vec{\oplus}}$ are held in reverse by $\leq^L_{\vec{\oplus}}$, so we'll only consider the latter.

Being a lexicographic order, $\leq^L_{\vec{\oplus}}$ inherits many properties from its component orders. For example, $\leq^L_{\vec{\oplus}}$ is a total order if and only if both $\leq^L_{\oplus_S}$ and $\leq^L_{\oplus_T}$ are total. We will consider the inheritance of lower bounds in detail. All properties of lower bounds hold dually for upper bounds.

Let $X \subseteq S$ and $Y \subseteq T$. We will show that (s, t) is a lower bound for $(X \times Y, \leq^L_{\vec{\oplus}})$ if and only if s is a lower bound for $(X, \leq^L_{\oplus_S})$ and t is a lower bound for $(Y, \leq^L_{\oplus_T})$.

Suppose s is a lower bound for X, and t is a lower bound for Y; that is,

$$\begin{aligned}
&\forall x \in X.\, s \leq^L_{\oplus_S} x, \text{ and} \\
&\forall y \in Y.\, t \leq^L_{\oplus_S} y
\end{aligned}$$

Then (s, t) is a lower bound for $X \times Y$: If (x, y) in $X \times Y$ then $s \leq^L_{\oplus_S} x$ and $t \leq^L_{\oplus_T} y$. We immediately get $s = x \Rightarrow t \leq^L_{\oplus_T} y$ too, and $(s, t) \leq^L_{\vec{\oplus}} (x, y)$ follows.

Conversely, suppose (s, t) is a lower bound for $X \times Y$. Let $x \in X$ and $y \in Y$. Since (s, t) is a lower bound, we have $(s, t) \leq^L_{\vec{\oplus}} (x, y)$, which means $s \leq^L_{\oplus_S} x$. Also $(s, t) \leq^L_{\vec{\oplus}} (s, y)$, which means $t \leq^L_{\oplus_T} y$. Therefore, s is a lower bound for X and t is a lower bound for Y.

We now show that either $\bigwedge(X \times Y) = (\bigwedge X, \bigwedge Y)$, or none of those three greatest lower bounds exist.

Suppose $s = \bigwedge X$ and $t = \bigwedge Y$, so in addition to their being lower bounds we have

$$\forall b \in S.\, (\forall x \in X.\, b \leq^L_{\oplus_S} x) \Rightarrow b \leq^L_{\oplus_S} s, \text{ and}$$
$$\forall b \in T.\, (\forall y \in Y.\, b \leq^L_{\oplus_T} y) \Rightarrow b \leq^L_{\oplus_T} t$$

Then $\bigwedge(X \times Y)$ exists and equals (s, t). We have already seen that (s, t) is a lower bound when s and t are lower bounds. It remains to show that (s, t) is greatest. Suppose (s', t') is a lower bound. Let $x \in X$ and $y \in Y$. Since (s', t') is a lower bound, we have both $(s', t') \leq^L_{\vec{\oplus}} (x, y)$ and $(s', t') \leq^L_{\vec{\oplus}} (s', y)$. From the first we infer $s' \leq^L_{\oplus_S} x$, and from the second we infer $t' \leq^L_{\oplus_T} y$. We have shown $(\forall x \in X.\, s' \leq^L_{\oplus_S} x)$ and $(\forall y \in Y.\, t' \leq^L_{\oplus_T} y)$. Therefore, because s and t are greatest, we get $s' \leq^L_{\oplus_S} s$ and $t' \leq^L_{\oplus_T} t$, and $(s', t') \leq^L_{\vec{\oplus}} (s, t)$ follows directly.

Conversely, suppose $\bigwedge(X \times Y) = (s, t)$ exists, so in addition to its being a lower bound we have

$$\forall b \in X \times Y.\, (\forall p \in X \times Y.\, b \leq^L_{\vec{\oplus}} p) \Rightarrow b \leq^L_{\vec{\oplus}} (s, t)$$

Then $\bigwedge X$ exists and equals s, and $\bigwedge Y$ exists and equals t. We have already seen that s and t are lower bounds when (s, t) is a lower bound. It remains to show that they are both greatest. Suppose s' is a lower bound for X. Then (s', t) is a lower bound for $X \times Y$. Since (s, t) is greatest we have $(s', t) \leq^L_{\vec{\oplus}} (s, t)$, which implies $s' \leq^L_{\oplus_S} s$, therefore s is greatest. Now suppose t' is a lower bound for Y. Then (s, t') is a lower bound for $X \times Y$. Since (s, t) is greatest we have $(s, t') \leq^L_{\vec{\oplus}} (s, t)$, which implies $t' \leq^L_{\oplus_T} t$, therefore t is greatest.