L11 Problem Set 2

Ramana Kumar (rk436)

November 4, 2010

1 $\mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge \mathrm{LK}(T) \implies \mathrm{LD}(S \vec{\times} T)$

Let $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T$. As in the case proved in the lecture slides, the terms to be equated are t_{rhs} and $t_1 \otimes_T t_{\mathrm{lhs}}$, where

$$(\cdot, t_{\mathrm{lhs}}) = (s_2, t_2) \vec{\oplus} (s_3, t_3), \text{ and}$$
$$(\cdot, t_{\mathrm{rhs}}) = (s_1 \otimes_S s_2, t_1 \otimes_T t_2) \vec{\oplus} (s_1 \otimes_S s_3, t_1 \otimes_T t_3)$$

There are four subcases to consider.

- $s_2 = s_2 \oplus_S s_3 = s_3$
  The proof for the first subcase proved in the slides also works here, since it did not make use of $\mathrm{LC}(S)$.

- $s_2 = s_2 \oplus_S s_3 \neq s_3$
  Here $t_{\mathrm{lhs}} = t_2$ and $s_1 \otimes_S s_2 = s_1 \otimes_S (s_2 \oplus_S s_3)$. Thus, using $\mathrm{LD}(S)$, we have $s_1 \otimes_S s_2 = (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3) =_? s_1 \otimes_S s_3$. If the $=_?$ is $\neq$, then $t_{\mathrm{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\mathrm{lhs}}$ as required. If the $=_?$ is $=$ then $t_{\mathrm{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)$. By $\mathrm{LD}(T)$, $t_{\mathrm{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3)$, and by $\mathrm{LK}(T)$ this equals $t_1 \otimes_T t_{\mathrm{lhs}}$ as required.

- $s_2 \neq s_2 \oplus_S s_3 = s_3$
  Here $t_{\mathrm{lhs}} = t_3$ and $s_1 \otimes_S s_3 = s_1 \otimes_S (s_2 \otimes_S s_3)$. Thus, using $\mathrm{LD}(S)$, we have $s_1 \otimes_S s_2 =_? (s_1 \otimes_S s_2) \oplus_S (s_2 \otimes_S s_3) = s_2 \otimes_S s_3$. If the $=_?$ is $\neq$, then $t_{\mathrm{rhs}} = t_1 \otimes_T t_3 = t_1 \otimes_T t_{\mathrm{lhs}}$ as required. If the $=_?$ is $=$ then $t_{\mathrm{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)$. By $\mathrm{LD}(T)$, $t_{\mathrm{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3)$, and by $\mathrm{LK}(T)$ this equals $t_1 \otimes_T t_{\mathrm{lhs}}$ as required.

- $s_2 \neq s_2 \oplus_S s_3 \neq s_3$
  Here $t_{\mathrm{lhs}} = \bar{0}_T$. The four possibilities for t_{rhs} are $(t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)$, $t_1 \otimes_T t_2$, $t_1 \otimes_T t_3$, and $\bar{0}_T$. The first case, using $\mathrm{LD}(T)$, can be written $t_1 \otimes_T (t_2 \oplus_T t_3)$. Thus in all cases, t_{rhs} is either $\bar{0}_T$ or $t_1 \otimes_T t$ for some t. But by $\mathrm{LK}(T)$, we know $t_1 \otimes_T t = t_1 \otimes_T t_{\mathrm{lhs}} = \bar{0}_T$, thus all cases give the required result.

  Note that we assume $\bar{0}_T$ is an annihilator for $\otimes_T$. The theorem should include an extra hypothesis, that if $\oplus_S$ is not selective (that is, this subcase may occur) then $\bar{0}_T$ exists and is an annihilator for $\otimes_T$.

2 $\mathrm{NLD}(S) \vee \mathrm{NLD}(T) \vee (\mathrm{NLC}(S) \wedge \mathrm{NLK}(T)) \implies \mathrm{NLD}(S \vec{\times} T)$

2.1 $\mathrm{NLD}(S)$

From $\mathrm{NLD}(S)$ we get witnesses a, b, and c such that $c \otimes_S (a \oplus_S b) \neq (c \otimes_S a) \oplus_S (c \otimes_S b)$. We can construct witnesses for $\mathrm{NLD}(S \vec{\times} T)$ as follows. Let $t \in T$. Consider $(c, t) \otimes ((a, t) \vec{\oplus} (b, t))$, which equals $(c,t) \otimes (a \oplus_S b, t') = (c \otimes_S (a \oplus_S b), t'')$ for some t' and t''. Now consider $((c,t) \otimes (a,t))\vec{\oplus}((c,t) \otimes (b,t))$, which equals $(c \otimes_S a, t \otimes_T t)\vec{\oplus}(c \otimes_S b, t \otimes_T t) = ((c \otimes_S a) \oplus_S (c \otimes_S b), t''')$ for some t'''. If these were equal, then their first components would be equal. So we would have $c \otimes_S (a \oplus_S b) = (c \otimes_S a) \oplus_S (c \otimes_S b)$. But we already know a, b, and c do not satisfy this property. Therefore (a,t), (b,t), and (c,t), for any t, serve as witnesses for NLD($S \vec{\times} T$).

Note we assumed T is not empty, and this condition should be added to the hypotheses of the theorem.

2.2 NLD(T)

From NLD(T) we get witnesses a, b, and c such that $c \otimes_T (a \oplus_T b) \neq (c \otimes_T a) \oplus_T (c \otimes_T b)$. We can construct witnesses for NLD($S \vec{\times} T$) as follows. Let $s \in S$. Consider $(s,c) \otimes ((s,a)\vec{\oplus}(s,b))$, which equals $(s,c) \otimes (s \oplus_S s, a \oplus_T b) = (s \otimes_S (s \oplus_S s), c \otimes_T (a \oplus_T b))$, as long as $s \oplus_S s = s$. Now consider $((s,c) \otimes (s,a))\vec{\oplus}((s,c) \otimes (s,b))$, which equals $(s \otimes_S s, c \otimes_T a)\vec{\oplus}(s \otimes_S s, c \otimes_T b) = ((s \otimes_S s) \oplus_S (s \otimes_S s), (c \otimes_T a) \oplus_T (c \otimes_T b))$, as long as $(s \otimes_S s) \oplus_S (s \otimes_S s) = s \otimes_S s$. If these were equal, then their second components would be equal. So we would have $c \otimes_T (a \oplus_T b) = (c \otimes_T a) \oplus_T (c \otimes_S b)$. But we already know a, b, and c do not satisfy this property. Therefore (s,a), (s,b), and (s,c), for any suitable s, serve as witnesses for NLD($S \vec{\times} T$).

Similarly to the previous case, we assumed S is not empty. We also need to assume that $\oplus_S$ satisfies $x \oplus_S x = x$ for both $x = s$ and $x = s \otimes_S s$ for some $s \in S$. This would be true if $\oplus_S$ were selective[1] or idempotent, or if $\bar{0}_S \in S$.

2.3 NLC(S) $\wedge$ NLK(T)

From NLC(S) we get witnesses a, b, and c such that $c \otimes_S a = c \otimes_S b$ and $a \neq b$. From NLK(T) we get witnesses d, e, and f such that $f \otimes_T d \neq f \otimes_T e$. We can construct witnesses for NLD($S \vec{\times} T$) as follows. We proceed by cases on $a \oplus_S b$ and $(f \otimes_T d) \oplus_T (f \otimes_T e)$.

- $a \oplus_S b = b \wedge (f \otimes_T d) \oplus_T (f \otimes_T e) = f \otimes_T d$ or $a \oplus_S b = a \wedge (f \otimes_T d) \oplus_T (f \otimes_T e) = f \otimes_T e$

[1]selectivity implies idempotence