L12 Mid-Term Test

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1. On Galois connections

(a) ($\Rightarrow$) In fact, g is monotone no matter whether f is monotone.

$$\boxed{\forall x \in Q, y \in Q.\, x \leq_{\mathsf{Q}} y \implies g(x) \leq_{\mathsf{P}} g(y)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{g(x) \leq_{\mathsf{P}} g(x)}\,\text{reflexivity}}{f(g(x)) \leq_{\mathsf{Q}} x}\,(1) \qquad \dfrac{}{x \leq_{\mathsf{Q}} y}\,\text{assumption}}{f(g(x)) \leq_{\mathsf{Q}} y}\,\text{transitivity}}{g(x) \leq_{\mathsf{P}} g(y)}\,(1)$$

($\Leftarrow$) Symmetrically, f is monotone. In the main theorem, instead of "iff" we have "and".

$$\boxed{\forall x \in P, y \in P.\, x \leq_{\mathsf{P}} y \implies f(x) \leq_{\mathsf{Q}} f(y)}$$

$$\dfrac{\dfrac{\dfrac{}{x \leq_{\mathsf{P}} y}\,\text{assumption} \qquad \dfrac{\dfrac{}{f(y) \leq_{\mathsf{Q}} f(y)}\,\text{reflexivity}}{y \leq_{\mathsf{P}} g(f(y))}\,(1)}{x \leq_{\mathsf{P}} g(f(y))}\,\text{transitivity}}{f(x) \leq_{\mathsf{Q}} f(y)}\,(1)$$

(b) **(1)**

$$\boxed{\forall S \subseteq Y, T \subseteq Y.\, S \subseteq T \implies f^{-1}[S] \subseteq f^{-1}[T]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{S \subseteq T}\,\text{assumption}}{\forall x \in X.\, f(x) \in S \implies f(x) \in T}\,\text{consequence of } \subseteq}{\forall x \in X.\, x \in f^{-1}[S] \implies x \in f^{-1}[T]}\,\text{definition of } f^{-1}}{f^{-1}[S] \subseteq f^{-1}[T]}\,\text{definition of } \subseteq$$

(2) Define $\exists_f$ by $\exists_f(S) = \{f(x) \mid x \in S\}$. To show $\exists_f \dashv f^{-1}[\_]$ is a Galois connection is to show

$$\boxed{\forall S \subseteq X, T \subseteq Y.\, \exists_f(S) \subseteq T \iff S \subseteq f^{-1}[T]}$$

$$
\dfrac{\dfrac{\dfrac{\overline{(\forall x \in S.\, f(x) \in T) \iff (\forall x \in S.\, f(x) \in T)}}{(\forall y.\, (\exists x \in S.\, y = f(x)) \implies y \in T) \iff (\forall x.\, x \in S \implies f(x) \in T)}\ \text{simplification}}{\{f(x) \mid x \in S\} \subseteq T \iff S \subseteq \{x \mid f(x) \in T\}}\ \text{definition of } \subseteq}{\exists_f(S) \subseteq T \iff S \subseteq f^{-1}[T]}\ \text{definitions of } \exists_f \text{ and } f^{-1}
$$

(3) Define $\forall_f$ by $\forall_f(T) = \{y \in Y \mid \forall x \in X.\, y = f(x) \implies x \in T\}$. To show $f^{-1}[\_] \dashv \forall_f$ is a Galois connection is to show

$$
\boxed{\forall S \subseteq Y, T \subseteq X.\ f^{-1}[S] \subseteq T \iff S \subseteq \forall_f(T)}
$$

$$
\dfrac{\dfrac{\dfrac{\overline{(\forall x.\, f(x) \in S \implies x \in T) \iff (\forall x.\, f(x) \in S \implies x \in T)}}{(\forall x.\, f(x) \in S \implies x \in T) \iff (\forall y.\, y \in S \implies \forall x.\, y = f(x) \implies x \in T)}\ \text{simplification}}{\{x \mid f(x) \in S\} \subseteq T \iff S \subseteq \{y \mid \forall x.\, y = f(x) \implies x \in T\}}\ \text{definition of } \subseteq}{f^{-1}[S] \subseteq T \iff S \subseteq \forall_f(T)}\ \text{definitions of } f^{-1} \text{ and } \forall_f
$$

2. On distributive categories

(a) Terminal object Let $1 \in \mathcal{S}$ be terminal, and let ${}_{S\times 1}! : S \times 1 \to 1$ be the unique morphism from $S \times 1$. We show that the S-action $(1, {}_{S\times 1}!)$ is terminal in S-act. Let $(A, \alpha) \in S$-act. Then ${}_A! : A \to 1$ in $\mathcal{S}$ gives a morphism $(A, \alpha) \to (1, {}_{S\times 1}!)$ in S-act:

$$
\begin{tikzcd}
S \times A \arrow[r, "\mathrm{id}_S \times {}_A!"] \arrow[d, "\alpha"'] & S \times 1 \arrow[d, "{}_{S\times 1}!"] \\
A \arrow[r, "{}_A!"'] & 1
\end{tikzcd}
$$

There is only one morphism $S \times A \to 1$ in $\mathcal{S}$, therefore ${}_A! \circ \alpha = {}_{S\times 1}! \circ (\mathrm{id}_S \times {}_A!)$ as required. Uniqueness of ${}_A!$ in S-act follows from its uniqueness in $\mathcal{S}$. Thus $(1, {}_{S\times 1}!)$ is terminal.

Binary products Let (A, α) and (B, β) be objects in S-act. The object

$$
(A, \alpha) \times (B, \beta) = (A \times B, \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle)
$$

is their product, with projections π_1 and π_2. To see that this is an object of S-act, observe that $A \times B$ is an object of $\mathcal{S}$ since $\mathcal{S}$ contains products, and check the types of the morphisms:

$$
\dfrac{\dfrac{\dfrac{\overline{\mathrm{id}_S : S \to S} \quad \overline{\pi_1 : A \times B \to A}}{\mathrm{id}_S \times \pi_1 : S \times (A \times B) \to S \times A} \quad \overline{\alpha : S \times A \to A}}{\alpha \circ (\mathrm{id}_S \times \pi_1) : S \times (A \times B) \to A} \quad \dfrac{\dfrac{\overline{\mathrm{id}_S : S \to S} \quad \overline{\pi_2 : A \times B \to B}}{\mathrm{id}_S \times \pi_2 : S \times (A \times B) \to S \times B} \quad \overline{\beta : S \times B \to B}}{\beta \circ (\mathrm{id}_S \times \pi_2) : S \times (A \times B) \to B}}{\langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle : S \times (A \times B) \to A \times B}
$$

We check that π_1 and π_2 are morphisms in S-act.

$$\begin{tikzcd}
S \times (A \times B) \arrow[r, "\mathrm{id}_S \times \pi_1"] \arrow[d, "\langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle"'] & S \times A \arrow[d, "\alpha"] \\
A \times B \arrow[r, "\pi_1"'] & A
\end{tikzcd}$$

$$\dfrac{\overline{\alpha \circ (\mathrm{id}_S \times \pi_1) = \alpha \circ (\mathrm{id}_S \times \pi_1)}}{\pi_1 \circ \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle = \alpha \circ (\mathrm{id}_S \times \pi_1)} \text{ universal property of } A \times B$$

$$\begin{tikzcd}
S \times (A \times B) \arrow[r, "\mathrm{id}_S \times \pi_2"] \arrow[d, "\langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle"'] & S \times B \arrow[d, "\beta"] \\
A \times B \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

$$\dfrac{\overline{\beta \circ (\mathrm{id}_S \times \pi_2) = \beta \circ (\mathrm{id}_S \times \pi_2)}}{\pi_2 \circ \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle = \beta \circ (\mathrm{id}_S \times \pi_2)} \text{ universal property of } A \times B$$

Finally, we check that $(A, \alpha) \times (B, \beta)$ satisfies the universal property for products.

$$\begin{tikzcd}
 & (A,\alpha) \times (B,\beta) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
(A,\alpha) & & (B,\beta) \\
 & (C,\gamma) \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, dotted, "\langle f,g \rangle"'] &
\end{tikzcd}$$

Suppose (C, γ) is another object with projections $f : (C, \gamma) \to (A, \alpha)$ and $g : (C, \gamma) \to (B, \beta)$. Since f and g are morphisms in S-act, the following diagrams commute.

$$\begin{tikzcd}
S \times C \arrow[r, "\mathrm{id}_S \times f"] \arrow[d, "\gamma"'] & S \times A \arrow[d, "\alpha"] \\
C \arrow[r, "f"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
S \times C \arrow[r, "\mathrm{id}_S \times g"] \arrow[d, "\gamma"'] & S \times B \arrow[d, "\beta"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

We can take the product of morphisms $\langle f, g \rangle : (C, \gamma) \to (A, \alpha) \times (B, \beta)$ in S-act to be the product $\langle f, g \rangle : C \to A \times B$ in $\mathcal{S}$. This is a morphism in S-act:

$$\begin{tikzcd}
S \times C \arrow[r, "\mathrm{id}_S \times \langle f,g \rangle"] \arrow[d, "\gamma"'] & S \times (A \times B) \arrow[d, "\langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle"] \\
C \arrow[r, "\langle f,g \rangle"'] & A \times B
\end{tikzcd}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle f\circ\gamma, g\circ\gamma\rangle = \langle f\circ\gamma, g\circ\gamma\rangle}}{\langle f\circ\gamma, g\circ\gamma\rangle = \langle \alpha\circ(\mathrm{id}_S\times f), \beta\circ(\mathrm{id}_S\times g)\rangle}\ f, g \text{ are morphisms in } S\text{-}\mathsf{act}}{\langle f\circ\gamma, g\circ\gamma\rangle = \langle \alpha\circ(\mathrm{id}_S\times(\pi_1\circ\langle f,g\rangle)), \beta\circ(\mathrm{id}_S\times(\pi_2\circ\langle f,g\rangle))\rangle}\ \text{universal property of } A\times B}{\langle f\circ\gamma, g\circ\gamma\rangle = \langle \alpha\circ((\mathrm{id}_S\circ\mathrm{id}_S)\times(\pi_1\circ\langle f,g\rangle)), \beta\circ((\mathrm{id}_S\circ\mathrm{id}_S)\times(\pi_2\circ\langle f,g\rangle))\rangle}\ \mathrm{id}_S \text{ is identity}}{\langle f\circ\gamma, g\circ\gamma\rangle = \langle \alpha\circ(\mathrm{id}_S\times\pi_1)\circ(\mathrm{id}_S\times\langle f,g\rangle), \beta\circ(\mathrm{id}_S\times\pi_2)\circ(\mathrm{id}_S\times\langle f,g\rangle)\rangle}\ \text{composition of direct products}}{\langle f,g\rangle\circ\gamma = \langle \alpha\circ(\mathrm{id}_S\times\pi_1), \beta\circ(\mathrm{id}_S\times\pi_2)\rangle\circ(\mathrm{id}_S\times\langle f,g\rangle)}\ \text{composition with product}
$$

The universal property diagram commutes because it commutes in $\mathcal{S}$: we only need consider the first component of each object since morphisms in S-act are morphisms in $\mathcal{S}$ on the first components, and composition in S-act is composition in $\mathcal{S}$. Similarly, the product of morphisms $\langle f, g\rangle$ is unique in S-act because it is unique in $\mathcal{S}$.

We have shown that S-act has a terminal object and a binary product for each pair of objects. It follows that S-act is cartesian.

(b) Suppose $\mathcal{S}$ is distributive. Then for any objects A, B, and C, the canonical map $[\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]$ is an isomorphism $(A\times B)+(A\times C) \simeq A\times(B+C)$, and for any object A, the canonical map $\langle !_A, \mathrm{id}_0\rangle$ is an isomorphism $0 \simeq A\times 0$. To show that S-act is distributive is to show that these maps exist and are isomorphisms in S-act. For the maps to exist, we must first show that S-act is bicartesian. We have seen that S-act is cartesian when $\mathcal{S}$ is cartesian. We proceed to show that S-act is cocartesian when $\mathcal{S}$ is distributive, and then that S-act is also distributive.

Initial object Let $0 \in \mathcal{S}$ be initial, and let $!_{S\times 0} : 0 \to S\times 0$ be the unique morphism to $S\times 0$. Since $\mathcal{S}$ is distributive, the map $!_{S\times 0}$ is an isomorphism and has an inverse. We show that the S-action $(0, (!_{S\times 0})^{-1})$ is initial in S-act. Let $(A,\alpha) \in S$-act. Then $!_A : 0 \to A$ in $\mathcal{S}$ gives a morphism $(0, (!_{S\times 0})^{-1}) \to (A,\alpha)$ in S-act:

$$
\begin{tikzcd}
S\times 0 \arrow[r, "\mathrm{id}_S\times !_A"] \arrow[d, "(!_{S\times 0})^{-1}"'] & S\times A \arrow[d, "\alpha"] \\
0 \arrow[r, "!_A"'] & A
\end{tikzcd}
$$

There is only one morphism $0 \to A$ in $\mathcal{S}$, therefore $\alpha\circ(\mathrm{id}_S\times !_A)\circ !_{S\times 0} = !_A$. Composing on the right with the inverse isomorphism, we obtain $\alpha\circ(\mathrm{id}_S\times !_A) = !_A\circ(!_{S\times 0})^{-1}$ as required. Uniqueness of $!_A$ in S-act follows from its uniqueness in $\mathcal{S}$. Thus $(0, (!_{S\times 0})^{-1})$ is initial.

Binary coproducts Let (A,α) and (B,β) be objects in S-act. The object

$$(A,\alpha)+(B,\beta) = (A+B, (\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1, \mathrm{id}_S\times\iota_2]^{-1})$$

is their sum, with injections ι_1 and ι_2. To see that this is an object of S-act, observe that $A+B$ is an object of $\mathcal{S}$ since $\mathcal{S}$ contains coproducts, and check the types of the morphisms:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{id}_S : S \to S} \quad \dfrac{}{\iota_1 : A \to A + B}}{\mathrm{id}_S \times \iota_1 : S \times A \to S \times (A+B)} \quad \dfrac{\dfrac{}{\mathrm{id}_S : S \to S} \quad \dfrac{}{\iota_2 : B \to A + B}}{\mathrm{id}_S \times \iota_2 : S \times B \to S \times (A+B)}}{[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] : (S \times A) + (S \times B) \to S \times (A+B)}}{[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} : S \times (A+B) \to (S \times A) + (S \times B)} \quad \dfrac{\dfrac{}{\alpha : S \times A \to A} \quad \dfrac{}{\beta : S \times B \to B}}{(\alpha + \beta) : (S \times A) + (S \times B) \to A + B}}{(\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} : S \times (A+B) \to A + B}$$

Finally, we check that $(A, \alpha) + (B, \beta)$ satisfies the universal property for coproducts.

$$\begin{tikzcd}
 & (A,\alpha)+(B,\beta) \arrow[dd, dotted, "{[f,g]}"] & \\
(A,\alpha) \arrow[ur, "\iota_1"] \arrow[dr, "f"'] & & (B,\beta) \arrow[ul, "\iota_2"'] \arrow[dl, "g"] \\
 & (C,\gamma) &
\end{tikzcd}$$

Suppose (C, γ) is another object with injections $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (C, \gamma 0$. Since f and g are morphisms in S-act, the following diagrams commute.

$$\begin{tikzcd}
S \times A \arrow[r, "\mathrm{id}_S \times f"] \arrow[d, "\alpha"'] & S \times C \arrow[d, "\gamma"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
S \times B \arrow[r, "\mathrm{id}_S \times g"] \arrow[d, "\beta"'] & S \times C \arrow[d, "\gamma"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

We can take the sum of morphisms $[f, g] : (A, \alpha) + (B, \beta) \to (C, \gamma)$ in S-act to be the sum $[f, g] : A + B \to C$ in S. This is a morphism in S-act:

$$\begin{tikzcd}
S \times (A+B) \arrow[r, "{\mathrm{id}_S \times [f,g]}"] \arrow[d, "{(\alpha+\beta)\circ[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1}}"'] & S \times C \arrow[d, "\gamma"] \\
A + B \arrow[r, "{[f,g]}"'] & C
\end{tikzcd}$$

Distributivity Let (A, α), (B, β), and (C, γ) be objects in S-act. Then $[\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]$ is a morphism in S-act:

$$\begin{tikzcd}
S \times ((A \times B) + (A \times C)) \arrow[r, "{\mathrm{id}_S \times [\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]}"] \arrow[d, "{(\alpha+\beta)\circ[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1}}"'] & S \times (A \times (B + C)) \arrow[d, "?"] \\
(A \times B) + (A \times C) \arrow[r, "{[\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]}"'] & A \times (B + C)
\end{tikzcd}$$

We check that ι_1 and ι_2 are morphisms in S-act.

$$\begin{tikzcd} S\times A \arrow[r, "\mathrm{id}_S\times\iota_1"] \arrow[d, "\alpha"'] & S\times(A+B) \arrow[d, "(\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]^{-1}"] \\ A \arrow[r, "\iota_1"'] & A+B \end{tikzcd}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\iota_1\circ\alpha=(\alpha+\beta)\circ\iota_1}\;\text{universal property of } (S\times A)+(S\times B)\text{[1]}}{\iota_1\circ\alpha=(\alpha+\beta)\circ\mathrm{id}_{(S\times A)+(S\times B)}\circ\iota_1}\;\mathrm{id}_{(S\times A)+(S\times B)}\text{ is identity}}{\iota_1\circ\alpha=(\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]^{-1}\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]\circ\iota_1}\;\text{composition of inverses}}{\iota_1\circ\alpha=(\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]^{-1}\circ(\mathrm{id}_S\times\iota_1)}\;\text{universal property of } (S\times A)+(S\times B)$$

$$\begin{tikzcd} S\times B \arrow[r, "\mathrm{id}_S\times\iota_2"] \arrow[d, "\beta"'] & S\times(A+B) \arrow[d, "(\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]^{-1}"] \\ B \arrow[r, "\iota_2"'] & A+B \end{tikzcd}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\iota_2\circ\beta=(\alpha+\beta)\circ\iota_2}\;\text{universal property of } (S\times A)+(S\times B)}{\iota_2\circ\beta=(\alpha+\beta)\circ\mathrm{id}_{(S\times A)+(S\times B)}\circ\iota_2}\;\mathrm{id}_{(S\times A)+(S\times B)}\text{ is identity}}{\iota_2\circ\beta=(\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]^{-1}\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]\circ\iota_2}\;\text{composition of inverses}}{\iota_2\circ\beta=(\alpha+\beta)\circ[\mathrm{id}_S\times\iota_1,\mathrm{id}_S\times\iota_2]^{-1}\circ(\mathrm{id}_S\times\iota_2)}\;\text{universal property of } (S\times A)+(S\times B)$$

3. On sections and regular, strong, and extremal monomorphisms

(a) section $\implies$ regular Let $m : X \to Y$ be a section of $s : Y \to X$, which means $s\circ m = \mathrm{id}_X$. Then m is an equaliser of id_Y and $m\circ s$, since

$$\dfrac{\dfrac{\dfrac{}{m=m}}{m=m\circ\mathrm{id}_X}\;\mathrm{id}_X\text{ is identity}}{\mathrm{id}_Y\circ m=m\circ s\circ m}\;\mathrm{id}_Y\text{ is identity, } s\circ m=\mathrm{id}_X$$

and given any $m' : Z \to Y$ satisfying $\mathrm{id}_Y\circ m' = m\circ s\circ m'$, we immediately have $m' = m\circ(s\circ m')$, since id_Y is the identity.

regular $\implies$ strong Let $m : X \to Y$ be a monomorphism, so $\forall u, x : U \to X.\ m\circ u = m\circ x \implies u = x$. Furthermore, let m be regular, so there exist $f, g : Y \to Z$ such that m satisfies the following universal property: $\forall v : V \to Y.\ f\circ v = g\circ v \implies \exists! d : V \to X.\ v = m\circ d$.

$$\begin{tikzcd} X \arrow[r, "m"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \\ V \arrow[u, dashed, "d"] \arrow[ur, "v"'] & & \end{tikzcd}$$

Now let $e : U \to V$ be an epimorphism, so $\forall a, b : V \to Z.\ a \circ e = b \circ e \implies a = b$. Also let u and v morphisms such that the square below, without the diagonal, commutes.

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "u"'] & V \arrow[d, "v"] \arrow[dl, dotted, "d"'] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

We first show that there is a unique morphism $d : V \to X$ that makes the lower triangle commute.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f \circ m = g \circ m}\ m \text{ equalises } f, g}{f \circ m \circ u = g \circ m \circ u} \qquad \dfrac{}{m \circ u = v \circ e}\ \text{square commutes}}{f \circ v \circ e = g \circ v \circ e}}{f \circ v = g \circ v}\ e \text{ is epi}}{\exists! d.\ v = m \circ d}\ m \text{ equalises } f, g$$

We now show the upper triangle also commutes, given this morphism d.

$$\dfrac{\dfrac{\dfrac{}{m \circ u = v \circ e}\ \text{square commutes} \qquad \dfrac{}{v = m \circ d}\ \text{lower triangle commutes}}{m \circ u = m \circ d \circ e}}{u = d \circ e}\ m \text{ is mono}$$

It follows that d is unique among morphisms that make both triangles commute. Therefore m is strong.

strong $\implies$ extremal Let $m : X \to Y$ be a strong monomorphism. Now let $e : X \to V$ be an epimorphism such that triangle on the left commutes.

$$\begin{tikzcd} & V \arrow[d, "v"] \\ X \arrow[ur, "e"] \arrow[r, "m"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\mathrm{id}_X"'] & V \arrow[d, "v"] \\ X \arrow[r, "m"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\mathrm{id}_X"'] & V \arrow[d, "v"] \arrow[dl, "d"'] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

From the commutativity of the triangle follows the commutativity of the square in the middle, because $m \circ \mathrm{id}_X = m$. Then, since m is strong, there exists a unique $d : V \to X$ such that both triangles on the right commute.

We get $e \circ d = \mathrm{id}_X$ from the commutativity of the upper triangle. Additionally,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v \circ e = m}\ \text{square commutes} \qquad \dfrac{}{v = m \circ d}\ \text{lower triangle commutes}}{m \circ d \circ e = m}}{m \circ d \circ e = m \circ \mathrm{id}_V}\ \mathrm{id}_V \text{ is identity}}{d \circ e = \mathrm{id}_V}\ m \text{ is mono}$$

Therefore e is an isomorphism, with inverse d.

4. On retractions, sections, pushouts, and coequalisers

(a) $\Longrightarrow$ Suppose the square below is a pushout.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "r"'] & C \arrow[d, "q"] \\ B \arrow[r, "p"'] & P \end{tikzcd}$$

Then,

$$\dfrac{\dfrac{\dfrac{\overline{p \circ r = q \circ g}\ \text{square commutes}}{p \circ r \circ s = q \circ g \circ s}}{p \circ \mathrm{id}_B = q \circ g \circ s}\ r \text{ retracts } s}{p = q \circ g \circ s}\ \mathrm{id}_B \text{ is identity}$$

$\Longleftarrow$