Presheaves

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1 Introduction

A $\mathbf{V}$-valued presheaf, P, on a category C is a functor $P : C^{\text{op}} \to \mathbf{V}$. The functor category $[C^{\text{op}}, \mathbf{V}]$ is also known as a presheaf category. Presheaf categories inherit much structure from $\mathbf{V}$, and this can sometimes be useful in reasoning about the source category C. In particular, if homs in C are objects in $\mathbf{V}$, then C embeds fully and faithfully in its presheaf category via the Yoneda embedding. And when $\mathbf{V}$ has small (co)limits, then so does the presheaf category.

We take $\mathbf{V} = \mathbf{Set}$ and $C = \mathbb{C}$ to be a locally small category, and therefore only consider set-valued presheaves where the Yoneda embedding applies. The presheaf category on $\mathbb{C}$ is denoted $\widehat{\mathbb{C}}$. In this setting, we define the embedding and prove Yoneda's lemma, then use the embedding in the course of demonstrating the structure available in $\widehat{\mathbb{C}}$. In particular, we show that $\widehat{\mathbb{C}}$ is complete and cocomplete (because $\mathbf{Set}$ is), and cartesian closed (because of the embedding). We then show that every presheaf P can be recovered as a colimit of representable presheaves, where being representable means coming via the embedding. Finally, we show that if $\mathbb{C}$ is small, then $\widehat{\mathbb{C}}$ is equivalent to the free cocompletion of $\mathbb{C}$, that is, the "best" category that has all colimits and is the codomain of a functor from $\mathbb{C}$.

2 The Yoneda Embedding

An embedding is a functor that is full and faithful. Both of these notions relate to homs, that is, collections of morphisms of the same type. A functor F is full if it is surjective on homs, which means every morphism $k : FX \to FY$ in the target category is the image of some morphism $g : X \to Y$ in the source, that is, $k = Fg$. A functor is faithful if it is injective on homs, which means if $Ff = Fg$ for morphisms $f, g : X \to Y$ in the source, then in fact $f = g$.

We want to embed the category $\mathbb{C}$ in the presheaf category $\widehat{\mathbb{C}}$. The functor that does this, $\boldsymbol{y} : \mathbb{C} \to \widehat{\mathbb{C}}$, therefore assigns a functor to every object in $\mathbb{C}$, and a natural transformation to every morphism in $\mathbb{C}$. The trick is to assign objects to hom functors; a hom functor for x sends an object y to the hom of morphisms from y to x, thereby capturing the substantial information about x available from the collections of x-valued morphisms. By assuming $\mathbb{C}$ is locally small, we ensure that these homs are sets, and thus that hom functors are suitable as outputs of $\boldsymbol{y}$.

Definition 1. The *contravariant hom functor* for an object $x \in \mathbb{C}$, written $\mathbb{C}(-, x) : \mathbb{C}^{\text{op}} \to \mathbf{Set}$ is given by $\mathbb{C}(y, x)$ on objects $y \in \mathbb{C}$ and $\mathbb{C}(f, x) = (- \circ f) : \mathbb{C}(v, x) \to \mathbb{C}(u, x)$ on morphisms $f : u \to v$ in $\mathbb{C}$. (This is a functor because $(- \circ g) \circ (- \circ f) = (- \circ (f \circ g))$.)

The *covariant hom functor* for x, written $\mathbb{C}(x, -) : \mathbb{C} \to \mathbf{Set}$, is defined dually: $\mathbb{C}(x, y)$ on objects y and $\mathbb{C}(x, f) = f \circ -$ on morphisms f.

Proposition 1. *For each morphism $f : x \to y$ in $\mathbb{C}$, the collection $\{\mathbb{C}(z, f)\}_{z\in\mathbb{C}}$ is a natural transformation from $\mathbb{C}(-, x)$ to $\mathbb{C}(-, y)$.*

Proof. The naturality condition for a morphism $g : u \to v$ in $\mathbb{C}$ is:

$$\begin{array}{ccc} \mathbb{C}(v,x) & \xrightarrow{\mathbb{C}(g,x)} & \mathbb{C}(u,x) \\ {\scriptstyle \mathbb{C}(v,f)}\downarrow & & \downarrow{\scriptstyle \mathbb{C}(u,f)} \\ \mathbb{C}(v,y) & \xrightarrow[\mathbb{C}(g,y)]{} & \mathbb{C}(u,y) \end{array}$$

$$\mathbb{C}(g, y) \circ \mathbb{C}(v, f) = (- \circ g) \circ (f \circ -) = f \circ - \circ g = (f \circ -) \circ (- \circ g) = \mathbb{C}(u, f) \circ \mathbb{C}(g, x)$$

□

Definition 2. The *Yoneda functor* $\boldsymbol{y} : \mathbb{C} \to \widehat{\mathbb{C}}$ is given by $\mathbb{C}(-, x)$ on objects $x \in \mathbb{C}$ and $\{\mathbb{C}(z, f)\}_{z\in\mathbb{C}}$ on morphisms $f \in \mathbb{C}$. (This is a functor because composition in a functor category is component-wise: $\{\mathbb{C}(z, f)\}_{z\in\mathbb{C}} \circ \{\mathbb{C}(z, g)\}_{z\in\mathbb{C}} = \{f \circ -\}_{z\in\mathbb{C}} \circ \{g \circ -\}_{z\in\mathbb{C}} = \{f \circ g \circ -\}_{z\in\mathbb{C}} = \{\mathbb{C}(z, f \circ g)\}_{z\in\mathbb{C}}$.)

We will see (Theorem 3) that the Yoneda functor is straightforwardly faithful. To show that it is full, we make use of a famous fact: the Yoneda lemma. It says that the map sending a natural transformation $\eta : \boldsymbol{y}\, x \Longrightarrow (F : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set})$ to the element $\eta_x(\mathrm{id}_x) \in Fx$ has an inverse, which establishes a bijection between the sets $\widehat{\mathbb{C}}(\boldsymbol{y}\, x, F) \cong Fx$. (In fact, the bijections as x ranges over $\mathrm{ob}\,\mathbb{C}$ behave like a natural isomorphism, but we do not prove this.)

Theorem 2 (Yoneda)**.** *For all presheaves $F \in \widehat{\mathbb{C}}$ and objects $x \in \mathbb{C}$, the map $\eta \mapsto \eta_x(\mathrm{id}_x)$ is a bijection between $\widehat{\mathbb{C}}(\boldsymbol{y} x, F)$ and Fx.*

Proof. Let $\eta : \boldsymbol{y}\, x \Longrightarrow F$, so $\eta_x : \mathbb{C}(x, x) \to Fx$ and hence $\eta_x(\mathrm{id}_x) \in Fx$. To show that this map has an inverse is to show that every element $a \in Fx$ determines a unique natural transformation μ^a that satisfies $\mu^a_x(\mathrm{id}_x) = a$. Let $a \in Fx$. Take $\mu^a = \{\lambda f.\, Ff(a)\}_{z\in\mathbb{C}}$. Observe that this has the right type: the argument to μ^a_z is a morphism $f \in \boldsymbol{y}\, x = \mathbb{C}(z, x)$, so $Ff : Fx \to Fz$, which means $Ff(a) \in Fz$ and thus $\mu^a : \boldsymbol{y}\, x \longrightarrow F$. Also $\mu^a_x(\mathrm{id}_x) = (F\,\mathrm{id}_x)(a) = \mathrm{id}_{Fx}(a) = a$. Now we show that μ^a is natural. For a morphism $g : u \to v$ in $\mathbb{C}^{\mathrm{op}}$, the naturality condition is

$$\begin{array}{ccc} \mathbb{C}(u,x) & \xrightarrow{\mathbb{C}(g,x)} & \mathbb{C}(v,x) \\ {\scriptstyle \mu^a_u}\downarrow & & \downarrow{\scriptstyle \mu^a_v} \\ Fu & \xrightarrow[Fg]{} & Fv \end{array}$$

Let $f \in \mathbb{C}(u, x)$, then $Fg(\mu^a_u(f)) = Fg(Ff(a)) = F(f \circ g)(a) = \mu^a_v(f \circ g) = \mu^a_v(\mathbb{C}(g, x)(f))$. Thus $\mu^a : \boldsymbol{y}\, x \Longrightarrow F$. Finally we show that μ^a is unique. Suppose $\mu' : \boldsymbol{y}\, x \Longrightarrow F$ and $\mu'_x(\mathrm{id}_x) = a$. Then for any $z \in \mathbb{C}$ and $f \in \mathbb{C}(z, x)$, we have $\mu'_z(\mathbb{C}(f, x)(\mathrm{id}_x)) = Ff(\mu'_x(\mathrm{id}_x))$ by naturality for f at $\mathrm{id}_x \in \mathbb{C}(x, x)$.[1] But $\mu'_z(f) = \mu'_z(\mathrm{id}_x \circ f) = \mu'_z(\mathbb{C}(f, x)(\mathrm{id}_x)) = Ff(\mu'_x(\mathrm{id}_x)) = Ff(a) = \mu^a_z(f)$, therefore $\mu' = \mu^a$. □

[1] We are taking $g = f$, $u = x$, $v = z$, and $f = \mathrm{id}_x$ in the diagram used to show that μ^a is natural, to see it as a naturality condition for μ'. This change, where the f that was considered as an object has become the g and is considered as a morphism, is the key to the lemma, and the reason to use homs.

Definition 3. Given a presheaf $F : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ an object $x \in \mathbb{C}$, and an object $a \in Fx$, let $\widetilde{a}$ be the unique natural transformation $\boldsymbol{y}\,x \Longrightarrow F$ such that $\widetilde{a}_x(\mathrm{id}_x) = a$. That is, $\widetilde{a} = \{\lambda f.\, Ff(a)\}_{z \in \mathbb{C}}$.

Theorem 3. *$\boldsymbol{y}$ is an embedding*

Proof. First, we show it is faithful. Suppose $\boldsymbol{y}\,f = \boldsymbol{y}\,g$ for morphisms $f, g : x \to y$ in $\mathbb{C}$. Since $(\boldsymbol{y}\,f)_x(\mathrm{id}_x) = \mathbb{C}(x, f)(\mathrm{id}_x) = f \circ \mathrm{id}_x = f$, and similarly $(\boldsymbol{y}\,g)_x(\mathrm{id}_x) = g$, it follows that $f = g$.

Second, we show it is full. Let $k : \boldsymbol{y}\,x \to \boldsymbol{y}\,y$ be a morphism in $\widehat{\mathbb{C}}$, that is $k : \mathbb{C}(-, x) \Longrightarrow \mathbb{C}(-, y)$ for some $x, y \in \mathbb{C}$. Then $k_x(\mathrm{id}_x) \in \mathbb{C}(x, y)$ is a morphism $x \to y$ in $\mathbb{C}$ so we may consider the natural transformation $\boldsymbol{y}(k_x(\mathrm{id}_x))$. As above, we have $(\boldsymbol{y}(k_x(\mathrm{id}_x)))_x(\mathrm{id}_x) = k_x(\mathrm{id}_x)$, but Theorem 2 says k is the only natural transformation with this property, so $k = \boldsymbol{y}(k_x(\mathrm{id}_x))$. □

3 Complete, Cocomplete, Closed

A category is complete if it has all small limits, cocomplete if it has all small colimits, and cartesian closed if it has finite products and function spaces. Products are a kind of limit, so a complete category is closed if it has function spaces.

A limit of a diagram (functor) $D : J \to C$ of shape J in a category C is an object $L \in C$ and a natural transformation $\psi : \kappa_L \Longrightarrow D$, that is universal. ($\kappa_L$ is the constant $J \to C$ functor given by $\kappa_L(x) = L$, $K_L(f) = \mathrm{id}_L$.) The data (L, ψ) is called a cone of D, with vertex L, and it is a limit if it is terminal in the category of cones of D, which means every other cone (L', ϕ') of D factors through (L, ϕ) via a unique morphism $L' \to L$ in C. A limit is small if the shape of the diagram is a small category, that is, if the the collection of morphisms of J is a set (it follows that the collection of objects is also a set, since every object has an identity morphism). Colimits are dual: the natural transformation goes from $D \Longrightarrow \kappa_L$, and a cocone is a colimit if it is initial in the category of cocones.

A function space (or exponential object) between two objects x, y in a category C with binary products is an object $(x \Rightarrow y)$ and a morphism $\epsilon : (x \Rightarrow y) \times x \to y$ in C such that for every morphism $g : z \times x \to y$ there exist a unique morphism $\widehat{g} : z \to (x \Rightarrow y)$ such that $\epsilon \circ (\widehat{g} \times \mathrm{id}_x) = g$. This is a universal property, but not one that can be expressed as a limit.

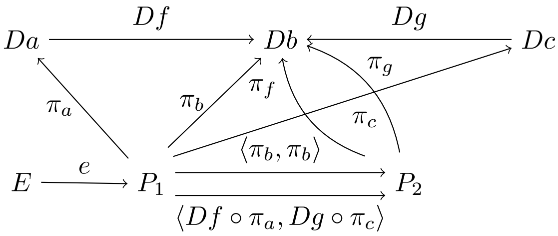
Two examples of limits are singled out for their importance: small products, and equalisers. A category with just these limits is in fact complete.

A product is a limit whose shape is a discrete category, that is, a category where all morphisms are identities. We write $\prod_{x \in J} Dx$ for the vertex of a product of the objects Dx, and reuse the letter π for the natural transformation, calling the components π_x projections. Given a collection $\{f_x\}_{x \in J}$ of morphisms from P to the objects Dx, we write $\langle f \rangle : P \to \prod_x Dx$ for the unique map given by the universal property. Dually, we write $\coprod_{x \in J} Dx$ for the vertex of a coproduct, ι_x for the components, called injections, of the natural transformation, and $[f]$ for the unique map out of a coproduct.

An equaliser is a limit of shape $\bullet \rightrightarrows \bullet$ (two objects, two morphisms plus identities). A cone of such a diagram can be represented by a single component, an incoming morphism on the left whose composites with the two parallel arrows are equal, because then the composite (either one) serves as the second required component. We write $\mathrm{Eq}(f, g)$ for the vertex of the cone equalising f and g, and reuse the letter e for the morphism $\mathrm{Eq}(f, g) \to \mathrm{dom}\, f$.

Theorem 4. *A category with small products and equalisers is complete. Dually, a category with small coproducts and coequalisers is cocomplete.*

Proof. Let C be a category with small products and equalisers. Let $D : J \to C$ be a diagram, where J is small. Then $S_1 = \{j \mid j \in \mathrm{ob}\, J\}$ and $S_2 = \{f \mid f \in \mathrm{mor}\, J\}$ are both sets, and therefore can be viewed as small discrete categories by adding (inconsequential) identity morphisms. Thus we may consider $P_1 = \prod_{j \in S_1} Dj$ and $P_2 = \prod_{f \in S_2} D(\mathrm{cod}\, f)$ because C has small products (and because $\mathrm{cod} : S_2 \to S_1$, and therefore $D \circ \mathrm{cod}$ too, is a functor). Using the universal property of P_2, we can construct two parallel morphisms $\langle \{\pi_{\mathrm{cod}\, f}\}_{f \in S_2} \rangle$ and $\langle \{Df \circ \pi_{\mathrm{dom}\, f}\}_{f \in S_2} \rangle$ in C, and then take their equaliser E. This is illustrated for the case when J has three objects and two non-identities in the following diagram in C.

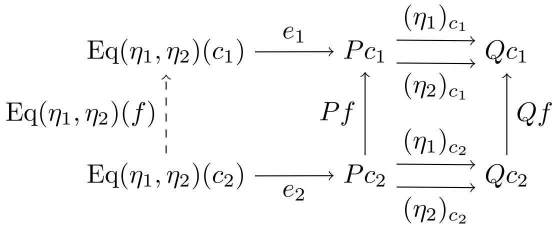


Here we have $\pi_f \circ \langle \pi_b, \pi_b \rangle = \pi_b = \pi_g \circ \langle \pi_b, \pi_b \rangle$, and $\pi_f \circ \langle Df \circ \pi_a, Dg \circ \pi_c \rangle = Df \circ \pi_a$ and $\pi_g \circ \langle Df \circ \pi_a, Dg \circ \pi_c \rangle = Dg \circ \pi_c$ since the products are cones, and $\langle \pi_b, \pi_b \rangle \circ e = \langle Df \circ \pi_a, Dg \circ \pi_c \rangle \circ e$ since the equaliser is a cone. Therefore, $Df \circ \pi_a \circ e = \pi_g \circ \langle Df \circ \pi_a, Dg \circ \pi_c \rangle \circ e = \pi_g \circ \langle \pi_b, \pi_b \rangle \circ e = \pi_b \circ e$, for example, which means $\pi_a \circ e$ and $\pi_b \circ e$ as components satisfy the naturality condition for f. In general, the limit of D is given by $(E, \{\pi_j \circ e\}_{j \in J})$, and it is universal because E is universal and forming a cone is equivalent to equalising the arrows between P_1 and P_2. □

For the following definition and theorem, we temporarily consider **V**-valued presheaves for an arbitrary complete category **V**. Completeness of a general presheaf category depends only on the completeness of the codomain of the presheaves, and we can use this generality to get cocompleteness for free (by considering $\mathbf{V}^{\mathrm{op}}$).

Definition 4. Let $D = \{P_j\}_{j \in J}$ be a family of (**V**-valued) presheaves indexed by a set J (where **V** is complete). The *product presheaf* Π_D is given by $\Pi_D(c) = \prod_{j \in J} P_j(c)$ and $\Pi_D(f) = \langle \{P_j(f) \circ \pi_j\}_{j \in J} \rangle$. The requirements of this as a functor, namely $\langle \{\pi_j\}_{j \in J} \rangle = \mathrm{id}_{\Pi_D(c)}$ and $\langle \{P_j(f) \circ P_j(g) \circ \pi_j\}_{j \in J} \rangle = \langle \{P_j(f) \circ \pi_j\}_{j \in J} \rangle \circ \langle \{P_j(g) \circ \pi_j\}_{j \in J} \rangle$, are instances of properties of products in general.

Let $\eta_1, \eta_2 : P \Longrightarrow Q$ be two morphisms in $\widehat{\mathbb{C}}$. The *equalising presheaf* $\mathrm{Eq}(\eta_1, \eta_2)$ is given by $\mathrm{Eq}(\eta_1, \eta_2)(c) = \mathrm{Eq}((\eta_1)_c, (\eta_2)_c)$, and $\mathrm{Eq}(\eta_1, \eta_2)(f)$ is the unique map given by the universal property of $\mathrm{Eq}(\eta_1, \eta_2)(\mathrm{dom}\, f)$. The action of $\mathrm{Eq}(\eta_1, \eta_2)$ on a morphism $f : c_1 \to c_2$ is illustrated below; that $Pf \circ e_2$ makes $(\eta_1)_{c_1}$ and $(\eta_2)_{c_1}$ equal follows from the naturality of η_1 and η_2.



$\mathrm{Eq}(\eta_1, \eta_2)$ is a functor because its action on morphisms produces a unique map, whose uniqueness condition (making a diagram like the above commute) is satisfied by the appropriate identity or composite. ¡more?¿

Theorem 5. *If* $\mathbb{C}$ *is small and* **V** *is complete then* $[\mathbb{C}^{\mathrm{op}}, \mathbf{V}]$ *is complete.*

Proof. In light of Theorem 4, it suffices to show that $[\mathbb{C}^{\mathrm{op}}, \mathbf{V}]$ has small products and equalisers.

Let $D : J \to [\mathbb{C}^{\mathrm{op}}, \mathbf{V}]$ be a diagram where J is small and discrete, so we can view D as a set-indexed family of presheaves, $D = \{P_j\}_{j \in J}$. Then $(\Pi_D, \{\{\pi_j\}_{c \in \mathbb{C}}\}_{j \in J})$ is a limit of D. There are no (non-identity) morphisms in J, so we only need to check that this pair is a cone of the right type and that it is terminal. For each $j \in J$, and each $c \in \mathbb{C}$ we have $\pi_j : \left(\prod_{j' \in J} P_{j'}(c)\right) \to P_j(c)$, that is $\pi_j : \Pi_D(c) \to Dj(c)$. And if $f : c_1 \to c_2$ in $\mathbb{C}$ then the following square commutes by definition of $\Pi_D(f)$ (as $\langle \{Dj(f) \circ \pi_j\}_{j \in J} \rangle$) and the universal property of the product $\Pi_D(c_1)$.

$$\begin{array}{ccc} \Pi_D(c_2) & \xrightarrow{\Pi_D(f)} & \Pi_D(c_1) \\ {\scriptstyle \pi_j}\downarrow & & \downarrow{\scriptstyle \pi_j} \\ Dj(c_2) & \xrightarrow[Dj(f)]{} & Dj(c_1) \end{array}$$

Therefore $\{\pi_j\}_{c \in \mathbb{C}} : \Pi_D \Longrightarrow Dj$ is a natural transformation of the right type. To see that the cone is limiting, consider another cone (X, η) of D. The unique morphism of cones $X \Longrightarrow \Pi_D$ is given by the unique morphism at each component $X(c) \to \Pi_D(c)$. ¡more?¿

Now consider two parallel morphisms, $\eta_1, \eta_2 : P \Longrightarrow Q$, in $\widehat{\mathbb{C}}$. Their equaliser is $\{e\}_{c \in \mathbb{C}} : \mathrm{Eq}(\eta_1, \eta_2) \Longrightarrow P$. That this is indeed a natural transformation, and equalises the morphisms universally, follows from that fact that each component is an equaliser in $\mathbf{V}$. ¡more?¿ □

Corollary 6. *If $\mathbb{C}$ is small and $\mathbf{V}$ is complete and cocomplete, then $[\mathbb{C}^{\mathrm{op}}, \mathbf{V}]$ is complete and cocomplete. In particular, if $\mathbb{C}$ is small then $\widehat{\mathbb{C}}$ is complete and cocomplete.*

Proof. Let $\mathbb{C}$ be a small category and let $\mathbf{V}$ be a complete and cocomplete category. By Theorem 5, we know $[\mathbb{C}^{\mathrm{op}}, \mathbf{V}]$ is complete. To show it is cocomplete is to show that $([\mathbb{C}^{\mathrm{op}}, \mathbf{V}])^{\mathrm{op}}$ is complete. It suffices to show that $([\mathbb{C}^{\mathrm{op}}, \mathbf{V}])^{\mathrm{op}} = [\mathbb{C}, \mathbf{V}^{\mathrm{op}}]$, because we are assuming $\mathbf{V}^{\mathrm{op}}$ is complete, we know $\mathbb{C}^{\mathrm{op}}$ is small since $\mathbb{C}$ is small, and $\mathbb{C} = (\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}$, so we can apply Theorem 5 again. A functor $F : \mathbb{C} \to \mathbf{V}^{\mathrm{op}}$ is essentially a contravariant functor from $\mathbb{C}$ to $\mathbf{V}$, and is thus the same as $F : \mathbb{C}^{\mathrm{op}} \to \mathbf{V}$, so the objects of $[\mathbb{C}, \mathbf{V}^{\mathrm{op}}]$ and $([\mathbb{C}^{\mathrm{op}}, \mathbf{V}])^{\mathrm{op}}$ are the same. A natural transformation $\eta : F \Longrightarrow (G : \mathbb{C} \to \mathbf{V}^{\mathrm{op}})$ has a component morphism $\eta_c : Fc \to Gc$ in $\mathbf{V}^{\mathrm{op}}$ for each object c in $\mathbb{C}$, which is equivalently a component morphism $\eta_c : Gc \to Fc$ in $\mathbf{V}$ for each object c in $\mathbb{C}^{\mathrm{op}}$, and has a naturality condition $Gf \circ \eta_{c_2} = \eta_{c_1} \circ Ff$ in $\mathbf{V}^{\mathrm{op}}$ for each $f : c_1 \to c_2$ in $\mathbb{C}$, which is equivalently a naturality condition $\eta_{c_2} \circ Gf = Ff \circ \eta_{c_1}$ in $\mathbf{V}$ for each $f : c_1 \to c_2$ in $\mathbb{C}^{\mathrm{op}}$. Therefore the morphisms of $[\mathbb{C}, \mathbf{V}^{\mathrm{op}}]$ and $([\mathbb{C}^{\mathrm{op}}, \mathbf{V}])^{\mathrm{op}}$ are the same, as required.

Now to show that $\widehat{\mathbb{C}}$ is complete and cocomplete, it is enough to show that **Set** is complete and cocomplete. But in **Set** Cartesian products give small products, restriction to a particular subset gives an equaliser, disjoint unions give small coproducts, and quotient by a particular equivalence relation gives a coequaliser. ¡more detail?¿ □

Before describing the function spaces in $\widehat{\mathbb{C}}$, we must look at adjunctions. In particular, we will prove and make use of the adjunction between the binary product and exponential functor for each object in a cartesian closed category.

Definition 5. An adjunction is a relationship between a pair of functors $F : C \to D$ and $G : D \to C$. We say F is left adjoint to G, written $F \dashv G$, if there is a bijective correspondence $\eta_{x,y}$ between the homs $D(Fx, y)$ and $C(x, Gy)$ for all objects $x \in C$ and $y \in D$, and η is natural in both x and y. Naturality here means that for each morphism $f : x_1 \to x_2$ in C and object y

in D, the following square commutes (in both directions)

$$\begin{array}{ccc} D(Fx_2, y) & \xrightarrow{- \circ Ff} & D(Fx_1, y) \\ \eta_{x_2,y} \updownarrow & & \updownarrow \eta_{x_1,y} \\ C(x_2, Gy) & \xrightarrow[- \circ f]{} & C(x_1, Gy) \end{array}$$

and for each morphism $g : y_1 \to y_2$ in D and object x in C, the following square commutes (in both directions)

$$\begin{array}{ccc} D(Fx, y_1) & \xrightarrow{g \circ -} & D(Fx, y_2) \\ \eta_{x,y_1} \updownarrow & & \updownarrow \eta_{x,y_2} \\ C(x, Gy_1) & \xrightarrow[Gg \circ -]{} & C(x, Gy_2) \end{array}$$

(Notice that we are essentially saying η is a natural isomorphism between two two-place hom functors. The domain of the functor $D(F-,=)$ would be straightforward to define: it is the product of categories $C \times D$. The codomain is complicated by size issues: it would be a category just like **Set**, but we do not want to restrict the notion of adjunction to locally small categories C and D.)

Definition 6. Let x be an object in a category C. If C has products, the *product functor*, denoted $-\times x$, is the endofunctor on C given by $c \times x$ on an object $c \in C$ and $f \times \mathrm{id}_x$ (that is, $\langle \pi_1 \circ f, \pi_2 \rangle$) on a morphism f in C. This is a functor because $\mathrm{id}_c \times \mathrm{id}_x = \mathrm{id}_{c \times x}$ and $(f \circ g) \times \mathrm{id}_x = (f \times \mathrm{id}_x) \circ (g \times \mathrm{id}_x)$ are properties of products in general.

If C also has function spaces, the *exponential functor*, denoted $(x \Rightarrow -)$, is the endofunctor on C given by $(x \Rightarrow c)$ on an object $c \in C$ and $\widehat{f \circ \epsilon}$ on a morphism f in C. ¡check type and functor properties¿

Proposition 7. *$(- \times c)$ is left adjoint to $(c \Rightarrow -)$ for each object c in a cartesian closed category.*

Proof. Let C be a cartesian closed category, and c an object in C. Define $\eta_{x,y} : C(x \times c, y) \to C(x, (c \Rightarrow y))$ by $f \mapsto \widehat{f}$ and its inverse by $g \mapsto \epsilon \circ (g \times \mathrm{id}_c)$. ¡check types¿ This is an inverse because $\epsilon \circ (\widehat{f} \times \mathrm{id}_c) = f$ by definition, and $\epsilon \circ \widehat{(g \times \mathrm{id}_c)}$ is the unique morphism $x \to (c \Rightarrow y)$ satisfying $\epsilon \circ (- \times \mathrm{id}_c) = \epsilon \circ (g \times \mathrm{id}_c)$ but g certainly satisfies that property so $\epsilon \circ \widehat{(g \times \mathrm{id}_c)} = g$. ¡check naturality¿ □

Proposition 8. *If $\mathbb{C}$ is small then $\widehat{\mathbb{C}}$ is locally small.*

Proof. Clearly, for all presheaves P and Q, the hom $\widehat{\mathbb{C}}(P, Q)$ is in bijection with

$$\left\{ \eta \in \prod_{c \in \mathbb{C}} (Pc \Rightarrow Qc) \;\middle|\; \forall f : a \to b \text{ in } \mathbb{C}.\, Qf \circ \eta_a = \eta_b \circ Pf \right\}$$

This is a set because the objects of $\mathbb{C}$ form a set, so the product is over a set, and the function space $(Pc \Rightarrow Qc)$ is an object in **Set**. □

Theorem 9. *If $\mathbb{C}$ is small, then $\widehat{\mathbb{C}}$ is closed.*

Proof. Let P and Q be presheaves in $\widehat{\mathbb{C}}$. The function space $(P \Rightarrow Q)$ may be defined by $(P \Rightarrow Q)\,(c) = \widehat{\mathbb{C}}(\boldsymbol{y}\,c \times P, Q)$ (which is a set by Proposition 8) and $(P \Rightarrow Q)\,(f) = - \circ (\boldsymbol{y}\,f \times \mathrm{id}_P)$. (The rationale for this action on objects is the Yoneda lemma, which suggests $(P \Rightarrow Q)\,(c) \cong \widehat{\mathbb{C}}(\boldsymbol{y}\,c, (P \Rightarrow Q))$, together with the adjunction in Proposition 7, which suggests $\widehat{\mathbb{C}}(\boldsymbol{y}\,c, (P \Rightarrow Q)) \cong \widehat{\mathbb{C}}(\boldsymbol{y}\,c \times P, Q)$.) The action on morphisms has the right type, because if $f : b \to a$ in $\mathbb{C}$ then $(P \Rightarrow Q)\,(f)$ precomposes a natural transformation $\eta : \boldsymbol{y}\,b \times P \Longrightarrow Q$ with $(\boldsymbol{y}\,f : \boldsymbol{y}\,a \to \boldsymbol{y}\,b) \times \mathrm{id}_P$, resulting in a natural transformation $\boldsymbol{y}\,a \times P \Longrightarrow Q$. Furthermore, we have defined a functor, since precomposition with $\boldsymbol{y}\,\mathrm{id}_c \times \mathrm{id}_P$ does nothing, and $- \circ (\boldsymbol{y}(f \circ g) \times \mathrm{id}_P) = - \circ ((\boldsymbol{y}\,f \circ \boldsymbol{y}\,g) \times \mathrm{id}_P) = - \circ (\boldsymbol{y}\,f \times \mathrm{id}_P) \circ (\boldsymbol{y}\,g \times \mathrm{id}_P) = (- \circ (\boldsymbol{y}\,g \times \mathrm{id}_P)) \circ (- \circ (\boldsymbol{y}\,f \times \mathrm{id}_P))$.

It remains to show that $(P \Rightarrow Q)$ satisfies the universal property of function spaces. The map $\epsilon : (P \Rightarrow Q) \times P \Longrightarrow Q$ is given by $\epsilon_c(\mu, x) = \mu_c(\mathrm{id}_c, x)$, which is natural, ultimately, because the argument $\mu : \boldsymbol{y}\,c \times P \Longrightarrow Q$ is natural. ¡show naturality in more detail?¿ Now let η be a natural transformation $R \times P \Longrightarrow Q$. The unique morphism $\widehat{\eta} : R \Longrightarrow (P \Rightarrow Q)$ is given by $\widehat{\eta}_c(y) = \{\lambda(f, x).\, \eta_{c'}((Rf)(y), x)\}_{c' \in \mathbb{C}}$. This makes the exponential diagram commute, because $\epsilon_c(\widehat{\eta}_c \times \mathrm{id}_P)(y, x) = \epsilon_c(\widehat{\eta}_c(y), x) = (\widehat{\eta}_c(y))_c(\mathrm{id}_c, x) = \eta_c((R\,\mathrm{id}_c)(y), x) = \eta_c(y, x)$, for all $c \in \mathbb{C}$, $y \in Rc$, and $x \in Pc$. ¡check unique¿ □

4 Presheaves are Colimits of Representables

A functor in $\widehat{\mathbb{C}}$ is representable if it is isomorphic to some hom functor $\boldsymbol{y}\,c$. Therefore a diagram $D : J \to \widehat{\mathbb{C}}$ is a diagram of representable presheaves (representables) if for all $x \in J$, we have $Dx \cong \boldsymbol{y}\,c$ for some $c \in C$. We will show in this section that every presheaf arises as the colimit of some diagram of representables, which means the representables in some sense "generate" the category $\widehat{\mathbb{C}}$. In fact, in the next section, we use this idea to show that $\widehat{\mathbb{C}}$ is the free cocompletion of $\mathbb{C}$.

What is the diagram of which a given presheaf P is a colimit? It is the Yoneda embedding restricted to the subcategory of objects whose hom functors have morphisms into P. We define the machinery to construct this subcategory below.

Definition 7. Given functors $T : X \to C$ and $S : Y \to C$, the *comma category*, denoted $(T \downarrow S)$, is given by:

- Objects are triples (x, y, η) with two objects $x \in X$ and $y \in Y$ and a morphism $\eta : Ta \to Sb$.
- Morphisms are pairs $(g, h) : (x, y, \eta) \to (x', y', \eta')$ of morphisms $g : x \to x'$ and $h : y \to y'$ such that $\eta' \circ Tg = Sh \circ \eta$.

The identity on (x, y, η) is $(\mathrm{id}_x, \mathrm{id}_y)$, and composition is given by $(g, h) \circ (g', h') = (g \circ g', h \circ h')$. (This is a category because $\eta \circ T(\mathrm{id}_x) = \eta = S(\mathrm{id}_y) \circ \eta$, and $((g, h) \circ (g', h')) \circ (g'', h'') = (g \circ g' \circ g'', h \circ h' \circ h'') = (g, h) \circ ((g', h') \circ (g'', h''))$.)

As a special case, when $Y = \bullet$, we obtain a generalized *slice category*, denoted (T/P) where $P = S(\bullet)$ is the chosen object in C. Since there is only one object in Y, we may simplify objects in (T/P) as pairs (x, η), and morphisms $g : (x, \eta) \to (x', \eta')$ as morphisms $g : x \to x'$ such that

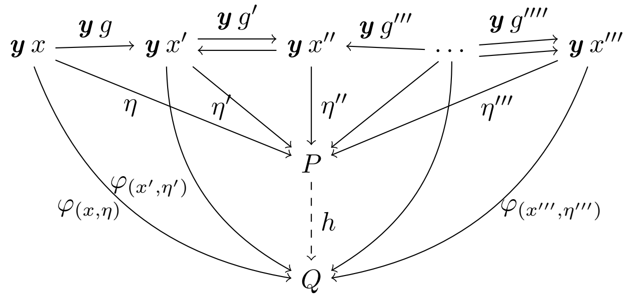
$$\begin{array}{ccc} Tx & \xrightarrow{\;Tg\;} & Tx' \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\eta'} & \\ & P & \end{array}$$

Definition 8. Given a presheaf $P \in \widehat{\mathbb{C}}$, the *representability diagram*, denoted $\Delta_P : (\boldsymbol{y}/P) \to \widehat{\mathbb{C}}$, is given by $\Delta_P(x,\eta) = \boldsymbol{y}\,x$ on objects and $\Delta_P(g) = \boldsymbol{y}\,g$. This is a functor because $\boldsymbol{y}$ is, and because slice category identities and composites are pointwise.

Theorem 10. *For all presheaves $P \in \widehat{\mathbb{C}}$, there exists a diagram $D : J \to \widehat{\mathbb{C}}$ of representables such that P is a (vertex of a) colimit of D.*

Proof. Let $P \in \widehat{C}$. Take $D = \Delta_P$, which is a diagram of representables by definition. To give a cocone with vertex P is to give a natural transformation $\psi : \Delta_P \Longrightarrow \kappa_P$. Take $\psi = \{\eta\}_{(x,\eta)\in(\boldsymbol{y}/P)}$. Then the naturality conditions for ψ match the conditions on morphisms in the slice category, so ψ is natural. It remains to show that this cocone is initial.

Let (Q,φ) be a cone of Δ_P. We must find a unique factorisation $h : (P,\psi) \to (Q,\varphi)$; in other words, the following diagram, showing a sketch of Δ_P and the two cones, must commute (in $\widehat{\mathbb{C}}$) for a unique h:



Take $h = \{\lambda a.\,(\varphi_{(x,\widetilde{a})})_x(\mathrm{id}_x)\}_{x\in\mathbb{C}}$. To check this is a morphism of cones from (P,ψ) to (Q,φ) is to check that it is a natural transformation $P \Longrightarrow Q$, and that it makes all triangles above involving h commute (the other triangles commute because the components of both cones are morphisms in $\widehat{\mathbb{C}}$ and therefore natural).

First, observe that if $a \in Px$, then $\widetilde{a} : \boldsymbol{y}\,x \Longrightarrow P$ by definition, so $(x,\widetilde{a})$ is an object in the slice category. Therefore $\varphi_{(x,\widetilde{a})} : \boldsymbol{y}\,x \Longrightarrow Q$, so $h_x(a) \in Qx$, thus $h : P \longrightarrow Q$. Naturality of h follows from naturality of $\varphi_{(x,\widetilde{a})}$.

Now look at $(h \circ \eta)_x(\mathrm{id}_x)$ for some $(x,\eta) \in (\boldsymbol{y}/P)$. We have $(h\circ\eta)_x(\mathrm{id}_x) = h_x(\eta_x(\mathrm{id}_x)) = \varphi_{(x,\widetilde{\eta_x(\mathrm{id}_x)})}(\mathrm{id}_x) = \varphi_{(x,\eta)}(\mathrm{id}_x)$, since $\eta = \widetilde{\eta_x(\mathrm{id}_x)}$ by definition, therefore, by Theorem 2, $h \circ \eta = \varphi_{(x,\eta)}$. Thus the triangles above commute.

Finally, h is unique because for each $(x,\eta) \in (\boldsymbol{y}/P)$, both η and $\varphi_{(x,\eta)}$ are determined by their action at component x on id_x. If a natural transformation $P \Longrightarrow Q$ is to make the diagram commute, its component at x must send $\eta_x(\mathrm{id}_x)$ to the same place $\varphi_{(x,\eta)}$ sends id_x. That is what h does. □

5 The Presheaf Category is the Free Cocomplete Category

A free construction gives an object some additional structure without making any unnecessary commitments. We can state this precisely in category theory by saying that any other object with the additional structure must factor through the free construction. Let us look at free cocompletions, where the additional structure is the existence of small colimits. A free cocompletition of C, is a cocomplete category $\overline{C}$ and a functor $C \xrightarrow{Y} \overline{C}$, such that for any cocomplete category and functor $C \xrightarrow{F} D$ there is a unique-up-to-isomorphism cocontinuous functor $\overline{F} : \overline{C} \to D$ such that $F = \overline{F} \circ Y$.

To show that our $\overline{F}$ is cocontinuous, we will make use of the following proposition, which says it suffices to show that $\overline{F}$ has a right adjoint.

Proposition 11. *Left adjoint functors are cocontinuous.*

Proof. Let $F \dashv (G : D \to C)$ be a pair of adjoint functors, and let (L, ψ) be a colimit of a diagram $J \to C$. ¡...¿ □

Theorem 12. *If* $\mathbb{C}$ *is small, then* $\mathbb{C} \overset{\boldsymbol{y}}{\hookrightarrow} \widehat{\mathbb{C}}$ *is a free cocompletion.*

Proof. By Theorem 6 we know that $\widehat{\mathbb{C}}$ is cocomplete. It remains to show that $\boldsymbol{y} : \mathbb{C} \to \widehat{\mathbb{C}}$ is universal.

Let $\mathbb{C} \overset{F}{\to} D$ be a functor to a cocomplete category. For each $P \in \widehat{\mathbb{C}}$, we have a diagram $(\boldsymbol{y}/P) \overset{U_P}{\to} \mathbb{C} \overset{F}{\to} D$, where U_P is the forgetful functor: $(x, \eta) \mapsto x$, $g \mapsto g$. The collection of objects of $(\boldsymbol{y}/P)$ is $\{(x, \eta) \mid \eta : \boldsymbol{y}\, x \Longrightarrow P\}$; Theorem 2 says this is $\bigcup_{x \in \mathbb{C}} \{(x, \widetilde{a}) \mid a \in Px\}$. Since $\mathbb{C}$ is small, the union is over a set, so $(\boldsymbol{y}/P)$ is small. Since D is cocomplete, $F \circ U_P$ has a colimit.

Let $(L_P, \psi_P) = \operatorname{colim}(F \circ U_P)$ for each $P \in \widehat{\mathbb{C}}$. For each $\mu : P \Longrightarrow Q$ in $\widehat{\mathbb{C}}$, there is a cocone of $F \circ U_P$ given by $(L_Q, \{(\psi_Q)_{(x, \mu \circ \eta)}\}_{(x,\eta) \in (\boldsymbol{y}/P)})$. To see that this is a cocone, observe that $(x, \mu \circ \eta) \in (\boldsymbol{y}/Q)$, because $\boldsymbol{y}\, x \overset{\eta}{\Longrightarrow} P \overset{\mu}{\Longrightarrow} Q$, and $\psi_Q : (F \circ U_Q) \Longrightarrow (\kappa_{L_Q} : (\boldsymbol{y}/Q) \to D)$, so $(\psi_Q)_{(x, \mu \circ \eta)} : Fx \to L_Q$, and the collection is natural because ψ_Q is natural. Since L_P is the vertex of a colimit, there is a unique map $\dot{\mu} : L_P \to L_Q$ in D factorising (L_Q, ψ_Q).

Take $\overline{F}$ to be the functor defined by $\overline{F}(P) = L_P$ and $\overline{F}(\mu) = \dot{\mu}$. This is a functor because post-composition always factorises a cocone, so if $P \overset{\mu}{\Longrightarrow} Q \overset{\mu'}{\Longrightarrow} R$, then $\dot{\mu}' \circ \dot{\mu}$ factorises (L_R, ψ_R) through (L_P, ψ_P), but $\dot{(\mu' \circ \mu)}$ is the unique map that does so, so they must be equal. It remains to show that $\overline{F}$ factorises F through $\boldsymbol{y}$, is cocontinuous, and is unique up to isomorphism.

We first observe that $(Fx, \{F(\eta_w(\mathrm{id}_w))\}_{(w,\eta) \in (\boldsymbol{y}/\boldsymbol{y}\, x)})$ is a colimit of $F \circ U_{\boldsymbol{y}\, x}$ for all $x \in \mathbb{C}$. It is a cocone: given any $f : (w, \eta) \to (w', \eta')$ we have $\eta' \circ \boldsymbol{y}\, f = \eta$ by definition of $(\boldsymbol{y}/\boldsymbol{y}\, x)$, therefore $\eta_w(\mathrm{id}_w) = \eta'_w((\boldsymbol{y}\, f)_w(\mathrm{id}_w))$, and we also have $(\boldsymbol{y}\, f)_w(\mathrm{id}_w) = f$ by Theorem 3 and $\eta'_w(\mathrm{id}_{w'} \circ f) = \eta'_{w'}(\mathrm{id}_{w'}) \circ f$ by naturality of η' hence $\eta_w(\mathrm{id}_w) = \eta'_{w'}(\mathrm{id}_{w'}) \circ f$ and thus $F(\eta_w(\mathrm{id}_w)) = F(\eta'_{w'}(\mathrm{id}_{w'})) \circ Ff$, but that is exactly the naturality condition for our cocone. It is colimiting: ¡check colimiting¿

Now given $f : x \to y$, the unique map from $\operatorname{colim}(F \circ U_{yx})$ to $\operatorname{colim}(F \circ U_{yy})$ has type $Fx \to Fy$ and satisfies $- \circ F(\eta_w(\mathrm{id}_w)) = F((\boldsymbol{y}\, f \circ \eta)_w(\mathrm{id}_w))$. Clearly it must be Ff. Thus we have $\overline{F}(\boldsymbol{y}\, x) = L_{\boldsymbol{y}\, x} = Fx$ for all objects x in $\mathbb{C}$ and $\overline{F}(\boldsymbol{y}\, f) = \dot{\boldsymbol{y}\, f} = Ff$ for all morphisms f in $\mathbb{C}$, therefore $F = \overline{F} \circ \boldsymbol{y}$.

¡show cocontinuous, by showing has right adjoint¿ ¡check uniqueness¿ □