L12 Test

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Q1.2

(i) Recall the (contravariant) Yoneda lemma, which says that for all functors $F : \mathbb{C}^{\text{op}} \to \boldsymbol{Set}$ and objects $A \in \mathbb{C}$, the set of natural transformations from the contravariant hom-functor h_A to F is in bijection with the set FA. Specifically, a natural transformation $\alpha : h_A \Rightarrow F$ is uniquely determined by $\alpha_A(\text{id}_A) \in FA$.

(only if) Suppose φ is a monomorphism in $\widehat{\mathbb{C}}$. This means that

$$\forall \alpha, \beta : Z \Rightarrow X.\ \varphi \circ \alpha = \varphi \circ \beta \implies \alpha = \beta$$

We now show that φ_c is injective for all objects c in $\mathbb{C}$. Let c be an object in $\mathbb{C}$. Let $x_1, x_2 \in Xc$ and suppose $\varphi_c(x_1) = \varphi_c(x_2)$. Now to show that φ is injective is to show that $x_1 = x_2$. Consider the natural transformations $\alpha : h_c \Rightarrow X$ and $\beta : h_c \Rightarrow X$ uniquely determined by setting $\alpha_c(\text{id}_c) = x_1$ and $\beta_c(\text{id}_c) = x_2$. The composites $\varphi \circ \alpha$ and $\varphi \circ \beta$ are both natural transformations from h_c to Y. We can show that they are equal by showing that they agree on id_c at component c.

$$\begin{aligned}(\varphi \circ \alpha)_c(\text{id}_c) = (\varphi_c \circ \alpha_c)(\text{id}_c) &= \varphi_c(\alpha_c(\text{id}_c)) \\ &= \varphi_c(x_1) = \varphi_c(x_2) \\ &= \varphi_c(\beta_c(\text{id}_c)) = (\varphi_c \circ \beta_c)(\text{id}_c) = (\varphi \circ \beta)_c(\text{id}_c)\end{aligned}$$

Therefore, by the Yoneda lemma, $\varphi \circ \alpha = \varphi \circ \beta$, and then since φ is mono we have $\alpha = \beta$. In particular $\alpha_c(\text{id}_c) = \beta_c(\text{id}_c)$, or equivalently $x_1 = x_2$. Therefore φ_c is injective.

(if) Suppose φ_c is injective for all $c \in \mathbb{C}$. This means that

$$\forall x_1, x_2 \in Xc.\ \varphi_c(x_1) = \varphi_c(x_2) \implies x_1 = x_2$$

We now show that φ is a monomorphism. Let α and β be natural transformations from $Z : \mathbb{C}^{\text{op}} \to \boldsymbol{Set}$ to X. Suppose $\varphi \circ \alpha = \varphi \circ \beta$. Now to show that φ is mono is to show that $\alpha = \beta$. Let c be an object in $\mathbb{C}$ and let $z \in Zc$. Now to show that $\alpha = \beta$ is to show that $\alpha_c(z) = \beta_c(z)$. Since φ_c is injective, it suffices to show $\varphi_c(\alpha_c(z)) = \varphi_c(\beta_c(z))$. But this follows from the assumption that $\varphi \circ \alpha = \varphi \circ \beta$.

(ii) (if) Suppose φ_c is surjective for all $c \in \mathbb{C}$. This means that

$$\forall y \in Yc.\ \exists x \in Xc.\ y = \varphi_c(x)$$

We now show that φ is an epimorphism. Let α and β be natural transformations from Y to $Z : \mathbb{C}^{\text{op}} \to \boldsymbol{Set}$. Suppose $\alpha \circ \varphi = \beta \circ \varphi$. Now to show that φ is epi is to show that

$\alpha = \beta$. Let c be an object in $\mathbb{C}$ and let $y \in Yc$. Now to show that $\alpha = \beta$ is to show that $\alpha_c(y) = \beta_c(y)$. Since φ_c is surjective, there exists an x such that $y = \varphi_c(x)$, so it suffices to show $\alpha_c(\varphi_c(x)) = \beta_c(\varphi_c(x))$. But this follows from the assumption that $\alpha \circ \varphi = \beta \circ \varphi$.

(only if) Suppose φ is an epimorphism in $\widehat{\mathbb{C}}$. This means that

$$\forall \alpha, \beta : Y \Rightarrow Z.\ \alpha \circ \varphi = \beta \circ \varphi \implies \alpha = \beta$$

We now show that φ_c is surjective for all objects c in $\mathbb{C}$. Let c be an object in $\mathbb{C}$, and let $y \in Yc$. Now to show that φ_c is surjective is to show that there exists an $x \in Xc$ such that $y = \varphi_c(x)$.

For each object $C \in \mathbb{C}$, define a predicate P_C by

$$\forall a \in YC.\ P_C(a) \iff \exists x \in XC.\ a = \varphi_C(x)$$

We have

$$\forall f : B \to A, a \in YA.\ P_A(a) \implies P_B((Yf)(a)) \tag{1}$$

because if $a = \varphi_A(x)$ then $(Yf)(a) = (Yf)(\varphi_A(x)) = \varphi_B((Xf)(x))$ by naturality of φ.

Define a functor $Z : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Set}$ as follows: $ZA = YA \times \{0, 1, 2\}$ and

$$Z(f : B \to A)(a, n) = \left((Yf)(a), \begin{cases} 0 & P_B((Yf)(a)) \wedge \neg P_A(a) \\ n & \text{otherwise} \end{cases}\right)$$

If $f : B \to A$, then $Zf : ZA \to ZB$ as required since $Yf : YA \to YB$ and if $n \in \{0, 1, 2\}$ then so are 0 and n. To confirm that Z is a functor is to show that $Z(\mathrm{id}_A) = \mathrm{id}_{ZA}$ for all objects $A \in \mathbb{C}$, and that $Z(f \circ g) = (Zg) \circ (Zf)$ for all morphisms $f : B \to A$ and $g : C \to B$ in $\mathbb{C}$. On an identity arrow, we have $Z(\mathrm{id}_A)(a, n) = \left((Y(\mathrm{id}_A))(a), \begin{cases} 0 & P_A((Y(\mathrm{id}_A))(a)) \wedge \neg P_A(a) \\ n & \text{otherwise} \end{cases}\right)$, which equals $\left(\mathrm{id}_{YA}(a), \begin{cases} 0 & P_A(\mathrm{id}_{YA}(a)) \wedge \neg P_A(a) \\ n & \text{otherwise} \end{cases}\right)$ since Y is a functor, and this equals (a, n) since $P_A(a)$ and $\neg P_A(a)$ cannot both hold. Therefore $Z(\mathrm{id}_A) = \mathrm{id}_{ZA}$. On a composite, we have $(Z(f \circ g))(a, n) = \left((Y(f \circ g))(a), \begin{cases} 0 & P_C((Y(f \circ g))(a)) \wedge \neg P_A(a) \\ n & \text{otherwise} \end{cases}\right)$, whereas $Zg((Zf)(a, n)) = \left((Yg)((Yf)(a)), \begin{cases} 0 & P_C((Yg)((Yf)(a)) \wedge \neg P_B((Yf)(a)) \\ 0 & P_B((Yf)(a)) \wedge \neg P_A(a) \\ n & \text{otherwise} \end{cases}\right)$, which equals $\left((Y(f \circ g))(a), \begin{cases} 0 & P_C((Y(f \circ g))(a)) \wedge \neg P_B((Yf)(a)) \\ 0 & P_B((Yf)(a)) \wedge \neg P_A(a) \\ n & \text{otherwise} \end{cases}\right)$ since Y is a functor. Of $(Z(f \circ g))(a, n)$ and $Zg((Zf)(a, n))$ the first components are equal, and for the second components to be equal it suffices that

$$Q_3 \wedge \neg Q_1 \iff (Q_3 \wedge \neg Q_2) \vee (Q_2 \wedge \neg Q_1)$$

where $Q_1 = P_A(a)$, $Q_2 = P_B((Yf)(a))$, and $Q_3 = P_C((Y(f \circ g))(a))$. By (1) we know $Q_1 \implies Q_2$ and $Q_2 \implies Q_3$ (again using functoriality of Y on Q_3). Therefore if Q_1 is true, then Q_2 is also true and both sides of the equation are false. So suppose $\neg Q_1$, which leaves us to show

$$Q_3 \iff (Q_3 \wedge \neg Q_2) \vee Q_2$$

Now if Q_2 is true, then Q_3 is also true and both sides are true, whereas if Q_2 is false, then the equation simplifies to $Q_3 \iff Q_3$ and both sides are equal. Therefore $(Z(f \circ g))(a, n) = Zg((Zf)(a, n))$, which means $Z(f \circ g) = (Zg) \circ (Zf)$ and hence Z is a functor.

For each object $A \in \mathbb{C}$ define two functions $YA \to ZA$ as follows:

$$\alpha_A(a) = \left(a, \begin{cases} 0 & P_A(a) \\ 1 & \text{otherwise} \end{cases}\right) \qquad \beta_A(a) = \left(a, \begin{cases} 0 & P_A(a) \\ 2 & \text{otherwise} \end{cases}\right)$$

We can show that the collections α and β are natural transformations. Let $f : B \to A$ be a morphism in $\mathbb{C}$. The naturality conditions are

$$\begin{array}{ccc} YA & \xrightarrow{Yf} & YB \\ {\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_B} \\ ZA & \xrightarrow[Zf]{} & ZB \end{array} \qquad \begin{array}{ccc} YA & \xrightarrow{Yf} & YB \\ {\scriptstyle\beta_A}\downarrow & & \downarrow{\scriptstyle\beta_B} \\ ZA & \xrightarrow[Zf]{} & ZB \end{array}$$

Let $a \in YA$. Suppose $P_A(a)$, whence (by (1)) $P_B((Yf)(a))$. Then

$$(Zf)(\alpha_A(a)) = (Zf)(a, 0) = ((Yf)(a), 0) = \alpha_B((Yf)(a))$$

Suppose instead $\neg P_A(a)$, and suppose $P_B((Yf)(a))$. Then

$$(Zf)(\alpha_A(a)) = (Zf)(a, 1) = ((Yf)(a), 0) = \alpha_B((Yf)(a))$$

Finally, suppose $\neg P_B((Yf)(a))$ as well as $\neg P_A(a)$. Then

$$(Zf)(\alpha_A(a)) = (Zf)(a, 1) = ((Yf)(a), 1) = \alpha_B((Yf)(a))$$

Therefore $(Zf) \circ \alpha_A = \alpha_B \circ (Yf)$ as required, and the same reasoning applies to β (with 2 in the second component instead of 1). Hence α and β are natural transformations $Y \Rightarrow Z$.

We now show that $\alpha \circ \varphi = \beta \circ \varphi$. Let A be an object in $\mathbb{C}$ and let $a \in XA$. Note that $P_A(\varphi_A(a))$ holds, with witness a. Therefore $\alpha_A(\varphi_A(a)) = (\varphi_A(a), 0) = \beta_A(\varphi_A(a))$. Hence $\alpha \circ \varphi = \beta \circ \varphi$.

Since φ is an epimorphism, it follows that $\alpha = \beta$. In particular, $\alpha_c(y) = \beta_c(y)$. Now if $P_c(y)$ is false, then $\alpha_c(y) = (y, 1)$ and $\beta_c(y) = (y, 2)$ but these are not equal. Therefore $P_c(y)$ must be true, which means there exists an $x \in Xc$ such that $y = \varphi_c(x)$. This finishes the proof that φ_c is surjective.

Q2.2

For natural transformations $\alpha : F_1 \Rightarrow F_2 : \mathcal{A} \to \mathcal{B}$ and $\beta : G_1 \Rightarrow G_2 : \mathcal{B} \to \mathcal{C}$ define the composition $\beta \circ \alpha : G_1F_1 \Rightarrow G_2F_2 : \mathcal{A} \to \mathcal{C}$ by $(\beta \circ \alpha)_A = \beta_{F_2A} \circ G_1\alpha_A$ for all $A \in \mathcal{A}$. To prove this definition yields a functor is to prove that it produces a morphism $(G_1F_1 \to G_2F_2)$ in $[\mathcal{A}, \mathcal{C}]$ for every morphism $(G_1, F_1) \to (G_2, F_2)$ in $[\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}]$, that it preserves identities, and that it preserves composition.

(well-defined) Take α and β as above, so $(\beta, \alpha) : (G_1, F_1) \to (G_2, F_2)$ is a morphism in $[\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}]$. Let A be an object in $\mathcal{A}$. Note that $\alpha_A : F_1A \to F_2A$ is a morphism in $\mathcal{B}$ so $G_1\alpha_A : G_1F_1A \to G_1F_2A$ is a morphism in $\mathcal{C}$, and F_2A is an object in $\mathcal{B}$ so $\beta_{F_2A} : G_1F_2A \to G_2F_2A$ is a morphism in $\mathcal{C}$; therefore our definition gives a component morphism from G_1F_1A to G_2F_2A in $\mathcal{C}$ at every object $A \in \mathcal{A}$. To prove that this collection of component morphisms is a morphism in $[\mathcal{A}, \mathcal{C}]$, we must show that it is natural. Naturality here means that for every morphism $f : X \to Y$ in $\mathcal{A}$, the following square commutes.

$$\begin{array}{ccc} G_1F_1X & \xrightarrow{G_1F_1f} & G_1F_1Y \\ \Big\downarrow{\scriptstyle (\beta\circ\alpha)_X} & & \Big\downarrow{\scriptstyle (\beta\circ\alpha)_Y} \\ G_2F_2X & \xrightarrow[G_2F_2f]{} & G_2F_2Y \end{array}$$

Since α and β are natural transformations, the following squares commute.

$$\begin{array}{ccc} F_1X & \xrightarrow{F_1f} & F_1Y \\ \Big\downarrow{\scriptstyle \alpha_X} & & \Big\downarrow{\scriptstyle \alpha_Y} \\ F_2X & \xrightarrow[F_2f]{} & F_2Y \end{array} \qquad \begin{array}{ccc} G_1F_2X & \xrightarrow{G_1F_2f} & G_1F_2Y \\ \Big\downarrow{\scriptstyle \beta_{F_2X}} & & \Big\downarrow{\scriptstyle \beta_{F_2Y}} \\ G_2F_2X & \xrightarrow[G_2F_2f]{} & G_2F_2Y \end{array}$$

From the square on the left (and functoriality of G_1) we deduce $G_1F_2f \circ G_1\alpha_X = G_1\alpha_Y \circ G_1F_1f$. Then

$$\begin{aligned} G_2F_2f \circ (\beta \circ \alpha)_X &= G_2F_2f \circ \beta_{F_2X} \circ G_1\alpha_X \\ &= \beta_{F_2Y} \circ G_1F_2f \circ G_1\alpha_X \\ &= \beta_{F_2Y} \circ G_1\alpha_Y \circ G_1F_1f = (\beta \circ \alpha)_Y \circ G_1F_1f \end{aligned}$$

as required.

(identities) Let (G, F) be an object in $[\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}]$. The identity morphism for (G, F) is given by $(\mathrm{id}_G, \mathrm{id}_F)$ where $\mathrm{id}_F : F \Rightarrow F : \mathcal{A} \to \mathcal{B}$ is the natural transformation with component id_{FA} at each object $A \in \mathcal{A}$, and analogously for id_G. We must show that $\mathrm{id}_G \circ \mathrm{id}_F = \mathrm{id}_{GF}$. Equivalently, for every object A in $\mathcal{A}$, we must show that $(\mathrm{id}_G \circ \mathrm{id}_F)_A = (\mathrm{id}_{GF})_A$, where $(\mathrm{id}_{GF})_A = \mathrm{id}_{GFA}$ by definition. This follows from our definition of the composite, and functoriality of G:

$$(\mathrm{id}_G \circ \mathrm{id}_F)_A = (\mathrm{id}_G)_{FA} \circ G(\mathrm{id}_F)_A = \mathrm{id}_{GFA} \circ G(\mathrm{id}_{FA}) = \mathrm{id}_{GFA} \circ \mathrm{id}_{GFA} = \mathrm{id}_{GFA}$$

(composition) Let $(\beta, \alpha) : (G_1, F_1) \to (G_2, F_2)$ and $(\delta, \gamma) : (G_2, F_2) \to (G_3, F_3)$ be morphisms in $[\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}]$. Their composition is given by $(\delta, \gamma) \circ (\beta, \alpha) = (\delta \circ \beta, \gamma \circ \alpha)$ where $\gamma \circ \alpha$ is the natural transformation with component $\gamma_A \circ \alpha_A$ at each object A in $\mathcal{A}$, and analogously for $\delta \circ \beta$. We must show that $(\delta \circ \beta) \circ (\gamma \circ \alpha) = (\delta \circ \gamma) \circ (\beta \circ \alpha)$. Equivalently, for every object A in $\mathcal{A}$, we must show that $((\delta \circ \beta) \circ (\gamma \circ \alpha))_A = ((\delta \circ \gamma) \circ (\beta \circ \alpha))_A$. First observe the naturality condition on β for the morphism $\gamma_A : F_2A \to F_3A$:

$$\begin{array}{ccc} G_1F_2A & \xrightarrow{G_1\gamma_A} & G_1F_3A \\ {\scriptstyle \beta_{F_2A}}\downarrow & & \downarrow{\scriptstyle \beta_{F_3A}} \\ G_2F_2A & \xrightarrow[G_2\gamma_A]{} & G_2F_3A \end{array}$$

Therefore $\beta_{F_3A} \circ G_1\gamma_A = G_2\gamma_A \circ \beta_{F_2A}$. Our desired equality follows thus:

$$\begin{aligned} &((\delta \circ \beta) \circ (\gamma \circ \alpha))_A \\ &= (\delta \circ \beta)_{F_3A} \circ G_1(\gamma \circ \alpha)_A \\ &= \delta_{F_3A} \circ \beta_{F_3A} \circ G_1(\gamma_A \circ \alpha_A) \\ &= \delta_{F_3A} \circ \beta_{F_3A} \circ G_1\gamma_A \circ G_1\alpha_A \\ &= \delta_{F_3A} \circ G_2\gamma_A \circ \beta_{F_2A} \circ G_1\alpha_A \\ &= (\delta \circ \gamma)_A \circ (\beta \circ \alpha)_A \\ &= ((\delta \circ \gamma) \circ (\beta \circ \alpha))_A \end{aligned}$$

Q3.1

Note that f is a functor between the preorders P and Q considered as categories, sending a morphism $p_1 \le p_2$ in P to the morphism $f(p_1) \le f(p_2)$ in Q. That $f : P \to Q$ has a right adjoint means for every object $q \in Q$, there is an object $g_q \in P$ and a morphism $f(g_q) \le q$ such that for all $p \in P$ with $f(p) \le q$ we have $p \le g_q$ (uniqueness of $p \le g_q$ is immediate because all morphisms are unique, and for the same reason $f(p) \le q$ must be the composition of $f(g_q) \le q$ after $f(p \le g_q)$).

(if) Suppose f preserves lubs. Let $q \in Q$. Take $g_q = \vee\{p \in P \mid f(p) \le q\}$. Since f preserves lubs we have $f(g_q) = \vee\{f(p) \mid p \in P \wedge f(p) \le q\}$. One of the defining conditions on $f(g_q)$ is (least) $\forall q' \in Q.\ (\forall p \in P.\ f(p) \le q \implies f(p) \le q') \implies f(g_q) \le q'$, from which $f(g_q) \le q$ follows directly. One of the defining conditions on g_q is (upper bound) $\forall p \in P.\ f(p) \le q \implies p \le g_q$, which is the remaining condition required for f to have a right adjoint.

(only if) Suppose f has a right adjoint. Thus we have

$$\forall q \in Q.\ f(g_q) \le q, \text{ and} \qquad \text{(ra1)}$$
$$\forall p \in P.\ f(p) \le q \implies p \le g_q \qquad \text{(ra2)}$$

Let $S \subseteq P$. For f to preserve lubs, we need $f(\vee S) = \vee\{f(s) \mid s \in S\}$, which means

$$\forall s \in S.\ f(s) \le f(\vee S), \text{ and}$$
$$\forall q \in Q.\ (\forall s \in S.\ f(s) \le q) \implies f(\vee S) \le q$$

For the first requirement, observe

$$\dfrac{\dfrac{}{\forall s \in S.\ s \le \vee S}\ (\vee S \text{ upper bound})}{\forall s \in S.\ f(s) \le f(\vee S)}\ (f \text{ monotone})$$

For the second requirement, let $q \in Q$ and observe

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall s \in S.\ f(s) \leq q}\ \text{(assumption)}}{\forall s \in S.\ s \leq g_q}\ \text{(ra2)}}{\vee S \leq g_q}\ (\vee S\ \text{least})}{f(\vee S) \leq f(g_q)}\ (f\ \text{monotone})}{f(\vee S) \leq q}\ \text{(ra1, transitivity)}
$$

Q3.2

For each object A in $\mathbb{C}$ define a functor $P_A : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Set}$ by $P_A(M) = \mathbb{C}(M \times c, A)$ for all objects M in $\mathbb{C}$ and $P_A(f) = - \circ (f \times \mathrm{id}_c)$ on morphisms f in $\mathbb{C}$. This is a functor because $P_A(\mathrm{id}_M) = - \circ (\mathrm{id}_M \times \mathrm{id}_c) = - \circ \mathrm{id}_{M\times c} = \mathrm{id}_{P_A(M)}$, and $P_A(f \circ g) = - \circ ((f \circ g) \times \mathrm{id}_c) = - \circ (f \times \mathrm{id}_c) \circ (g \times \mathrm{id}_c) = (- \circ (g \times \mathrm{id}_c)) \circ (- \circ (f \times \mathrm{id}_c)) = P_A(g) \circ P_A(f)$.

For each morphism $f : B \to A$ in $\mathbb{C}$ and object $X \in \mathbb{C}$, post-composition with f gives a morphism in $\boldsymbol{Set}$ from $P_A(X) \to P_B(X)$: if $g : X \times c \to A$ then $f \circ g : X \times c \to B$. Write $\widetilde{f}$ for this collection of morphisms over all objects X. Then $\widetilde{f} : P_A \Rightarrow P_B : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Set}$ is natural. Let $g : Y \to X$ be a morphism in $\mathbb{C}$. We must show $P_B(g) \circ \widetilde{f}_X = \widetilde{f}_Y \circ P_A(g)$. But $P_B(g) \circ \widetilde{f}_X = (- \circ (g \times \mathrm{id}_c)) \circ (f \circ -) = f \circ - \circ (g \times \mathrm{id}_c) = (f \circ -) \circ (- \circ (g \times \mathrm{id}_c)) = \widetilde{f}_Y \circ P_A(g)$.

Observe that the operation $\widetilde{-}$ is functorial. For $\widetilde{\mathrm{id}_A}$ is post-composition by id_A at each component, which clearly gives the identity natural transformation, and $\widetilde{f \circ g}$ has components $(\widetilde{f \circ g})_X = f \circ g \circ - = (f \circ -) \circ (g \circ -) = \widetilde{f}_X \circ \widetilde{g}_X$, so $\widetilde{f \circ g} = \widetilde{f} \circ \widetilde{g}$.

The functor $(-)^{\boldsymbol{y}c}$ has a right adjoint if for every object $Y \in \widehat{\mathbb{C}}$ there is an object $U_Y \in \widehat{\mathbb{C}}$ and a morphism $\epsilon_Y : (U_Y)^{\boldsymbol{y}c} \to Y$ such that for all objects $X \in \widehat{\mathbb{C}}$ with morphism $\alpha : X^{\boldsymbol{y}c} \to Y$ there is a unique morphism $\bar{\alpha} : X \to U_Y$ such that $\alpha = \epsilon_Y \circ (\bar{\alpha})^{\boldsymbol{y}c}$.

Let $Y \in \widehat{\mathbb{C}}$. Take U_Y to be the functor that acts as $U_Y(A) = \widehat{\mathbb{C}}(P_A, Y)$ on objects A in $\mathbb{C}$ and as $U_Y(f) = - \circ \widetilde{f}$ for all morphisms f in $\mathbb{C}$. This is a functor because $U_Y(\mathrm{id}_A) = - \circ \widetilde{\mathrm{id}_A} = \mathrm{id}_{U_Y(A)}$, since $\widetilde{\mathrm{id}_A}$ is the identity natural transformation, and $U_Y(f \circ g) = - \circ \widetilde{f \circ g} = - \circ \widetilde{f} \circ \widetilde{g} = (- \circ \widetilde{g}) \circ (- \circ \widetilde{f}) = U_Y(g) \circ U_Y(f)$.

Note that $(U_Y)^{\boldsymbol{y}c}$ is an exponential object in $\widehat{\mathbb{C}}$, so acts as $(U_Y)^{\boldsymbol{y}c}(A) = \widehat{\mathbb{C}}(\boldsymbol{y}A \times \boldsymbol{y}c, U_Y)$ on objects A in $\mathbb{C}$, and $(U_Y)^{\boldsymbol{y}c}(f) = - \circ (\boldsymbol{y}f \times \mathrm{id}_{\boldsymbol{y}c})$ on morphisms f in $\mathbb{C}$. At each object A in $\mathbb{C}$ we need a component morphism $(\epsilon_Y)_A : (U_Y)^{\boldsymbol{y}c}(A) \to YA$ in $\boldsymbol{Set}$. Define $(\epsilon_Y)_A(\beta) = ((\beta_{A\times c}(\pi_1, \pi_2))_A)(\mathrm{id}_{A\times c})$ for all $\beta : \boldsymbol{y}A \times \boldsymbol{y}c \Rightarrow U_Y : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Set}$. To show that this collection of components is natural is to show that for each morphism $f : N \to M$ in $\mathbb{C}$, the following naturality square commutes.

$$
\begin{array}{ccc}
(U_Y)^{\boldsymbol{y}c}(M) & \xrightarrow{(U_Y)^{\boldsymbol{y}c}(f)} & (U_Y)^{\boldsymbol{y}c}(N) \\
\Big\downarrow{\scriptstyle (\epsilon_Y)_M} & & \Big\downarrow{\scriptstyle (\epsilon_Y)_N} \\
YM & \xrightarrow[Yf]{} & YN
\end{array}
$$

Let $\beta : \boldsymbol{y}M \times \boldsymbol{y}c \Rightarrow U_Y$. From the naturality of β we have $U_Y(f \times \mathrm{id}_c) \circ \beta_{M\times c} = \beta_{N\times c} \circ (\boldsymbol{y}M \times \boldsymbol{y}c)(f \times \mathrm{id}_c)$, so in particular $U_Y(f \times \mathrm{id}_c)(\beta_{M\times c}(\pi_1, \pi_2)) = \beta_{N\times c}((\boldsymbol{y}M \times \boldsymbol{y}c)(f \times \mathrm{id}_c)(\pi_1, \pi_2))$.

Simplifying the right hand side, we have

$$\begin{aligned}
&\beta_{N\times c}((\boldsymbol{y}M \times \boldsymbol{y}c)(f \times \mathrm{id}_c)(\pi_1, \pi_2))\\
&= \beta_{N\times c}((\boldsymbol{y}M(f \times \mathrm{id}_c) \times \boldsymbol{y}c(f \times \mathrm{id}_c))(\pi_1, \pi_2))\\
&= \beta_{N\times c}(\boldsymbol{y}M(f \times \mathrm{id}_c)(\pi_1), \boldsymbol{y}c(f \times \mathrm{id}_c)(\pi_2))\\
&= \beta_{N\times c}(\pi_1 \circ (f \times \mathrm{id}_c), \pi_2 \circ (f \times \mathrm{id}_c))\\
&= \beta_{N\times c}(f \circ \pi_1, \mathrm{id}_c \circ \pi_2)\\
&= \beta_{N\times c}(f \circ \pi_1, \pi_2)
\end{aligned}$$

Thus $\beta_{N\times c}(f\circ\pi_1, \pi_2) = U_Y(f \times \mathrm{id}_c) \circ \beta_{M\times c}$. Additionally, from the naturality of $\beta_{M\times c}(\pi_1, \pi_2) : P_{M\times c} \Rightarrow Y$ we have $Yf \circ (\beta_{M\times c}(\pi_1, \pi_2))_M = (\beta_{M\times c}(\pi_1, \pi_2))_N \circ P_{M\times c}(f)$. Now observe that

$$\begin{aligned}
&(\epsilon_Y)_N(((U_Y)^{\boldsymbol{y}c}(f))(\beta))\\
&= (\epsilon_Y)_N(\beta \circ (\boldsymbol{y}f \times \mathrm{id}_{yc}))\\
&= ((((\beta \circ (\boldsymbol{y}f \times \mathrm{id}_{yc}))_{N\times c})(\pi_1, \pi_2))_N)(\mathrm{id}_{N\times c})\\
&= ((((\beta_{N\times c} \circ (\boldsymbol{y}f \times \mathrm{id}_{yc})_{N\times c}))(\pi_1, \pi_2))_N)(\mathrm{id}_{N\times c})\\
&= ((\beta_{N\times c}((\boldsymbol{y}f \times \mathrm{id}_{yc})_{N\times c}(\pi_1, \pi_2)))_N)(\mathrm{id}_{N\times c})\\
&= ((\beta_{N\times c}((\boldsymbol{y}f_{N\times c})(\pi_1), (\mathrm{id}_{yc})_{N\times c}(\pi_2)))_N)(\mathrm{id}_{N\times c})\\
&= ((\beta_{N\times c}(f \circ \pi_1, \pi_2))_N)(\mathrm{id}_{N\times c})\\
&= ((U_Y(f \times \mathrm{id}_c)(\beta_{M\times c}(\pi_1, \pi_2)))_N)(\mathrm{id}_{N\times c})\\
&= (((\beta_{M\times c}(\pi_1, \pi_2)) \circ \widetilde{f \times \mathrm{id}_c})_N)(\mathrm{id}_{N\times c})\\
&= ((\beta_{M\times c}(\pi_1, \pi_2))_N \circ (\widetilde{f \times \mathrm{id}_c})_N)(\mathrm{id}_{N\times c})\\
&= (\beta_{M\times c}(\pi_1, \pi_2))_N((\widetilde{f \times \mathrm{id}_c})_N(\mathrm{id}_{N\times c}))\\
&= (\beta_{M\times c}(\pi_1, \pi_2))_N((f \times \mathrm{id}_c) \circ \mathrm{id}_{N\times c})\\
&= (\beta_{M\times c}(\pi_1, \pi_2))_N(f \times \mathrm{id}_c)\\
&= (\beta_{M\times c}(\pi_1, \pi_2))_N(\mathrm{id}_{M\times c} \circ (f \times \mathrm{id}_c))\\
&= (\beta_{M\times c}(\pi_1, \pi_2))_N(P_{M\times c}(f)(\mathrm{id}_{M\times c}))\\
&= (Yf)(((\beta_{M\times c}(\pi_1, \pi_2))_M)(\mathrm{id}_{M\times c}))\\
&= (Yf)((\epsilon_Y)_M(\beta))
\end{aligned}$$

Therefore ϵ_Y is natural, so is a morphism in $\widehat{\mathbb{C}}$ as required.

Let $X \in \widehat{\mathbb{C}}$ and $\alpha : X^{\boldsymbol{y}c} \to Y$. The Yoneda lemma gives a bijection between $\widehat{\mathbb{C}}(\boldsymbol{y}A, X)$ and XA for all objects $A \in \mathbb{C}$, so given an element $x \in XA$ let $x^\sharp$ denote the corresponding natural transformation $\boldsymbol{y}A \Rightarrow X$. Furthermore, the Yoneda embedding $\boldsymbol{y}$ preserves limits, in particular products. Therefore $\boldsymbol{y}(B \times c) \cong \boldsymbol{y}B \times \boldsymbol{y}c$ for all objects $B \in \mathbb{C}$. Under this isomorphism, the component $\alpha_B : X^{\boldsymbol{y}c}(B) \to YB$, which takes a natural transformation $\boldsymbol{y}B \times \boldsymbol{y}c \Rightarrow X$, corresponds to function $(\alpha_B)^*$ that takes a natural transformation $\boldsymbol{y}(B \times c) \Rightarrow X$ instead, and this collection $\alpha^*_B = (\alpha_B)^*$ is natural from $\widehat{\mathbb{C}}(\boldsymbol{y}(- \times c), X)$ to Y. Let A be an object in $\mathbb{C}$ and let $x \in XA$. Then $\bar{\alpha}_A(x)$ must be a natural transformation $P_A \Rightarrow Y$. At each component $B \in \mathbb{C}$ define $(\bar{\alpha}_A(x))_B(g) = (\alpha_B)^*(x^\sharp \circ \boldsymbol{y}g)$ for all morphisms $g \in P_A(B) = \mathbb{C}(B \times c, A)$. To show that this collection of components is natural is to show that $(Yf)(\bar{\alpha}_A(x))_M(g) = (\bar{\alpha}_A(x))_N(P_A(f)(g))$ for all morphisms $f : N \to M$ in $\mathbb{C}$. By naturality of α^* we have $(Yf) \circ (\alpha_M)^* = (\alpha_N)^* \circ \widehat{\mathbb{C}}(\boldsymbol{y}(-\times$

$c), X)(f)$. Now observe that

$$
\begin{aligned}
&(Yf)(\bar{\alpha}_A(x))_M(g)\\
&= (Yf)((\alpha_M)^*(x^\sharp \circ \boldsymbol{y}g))\\
&= (\alpha_N)^*(\widehat{\mathbb{C}}(\boldsymbol{y}(- \times c), X)(x^\sharp \circ \boldsymbol{y}g))\\
&= (\alpha_N)^*(x^\sharp \circ \boldsymbol{y}g \circ \boldsymbol{y}(f \times \mathrm{id}_c))\\
&= (\alpha_N)^*(x^\sharp \circ \boldsymbol{y}(g \circ (f \times \mathrm{id}_c)))\\
&= (\bar{\alpha}_A(x))_N(g \circ (f \times \mathrm{id}_c))\\
&= (\bar{\alpha}_A(x))_N(P_A(f)(g))
\end{aligned}
$$

Therefore $(\bar{\alpha})_A$ is a function $XA \to U_Y(A)$ as required.

Furthermore, the collection $\bar{\alpha}$ is natural. Let $f : N \to M$ be a morphism in $\mathbb{C}$. The following naturality square must commute.

$$
\begin{array}{ccc}
XM & \xrightarrow{Xf} & XN\\
\Big\downarrow{\scriptstyle \bar{\alpha}_M} & & \Big\downarrow{\scriptstyle \bar{\alpha}_N}\\
U_Y(M) & \xrightarrow[U_Y(f)]{} & U_Y(N)
\end{array}
$$

Let $x \in XM$, $B \in \mathbb{C}$, and $g : B \times c \to N$ in $\mathbb{C}$. First note that $x^\sharp \circ \boldsymbol{y}f = ((Xf)(x))^\sharp : yN \Rightarrow X$ because they agree at component N on id_N; on the one hand $((Xf)(x))^\sharp_N(\mathrm{id}_N) = Xf(x)$ by definition and on the other hand $(x^\sharp \circ \boldsymbol{y}f)_N(\mathrm{id}_N) = x^\sharp_N((\boldsymbol{y}f)_N(\mathrm{id}_N)) = x^\sharp_N(f \circ \mathrm{id}_N) = x^\sharp_N(f) = Xf(x)$ by the Yoneda lemma. Now we reason as follows.

$$
\begin{aligned}
&(U_Y(f)(\bar{\alpha}_M(x)))_B(g)\\
&= (\bar{\alpha}_M(x) \circ \widetilde{f})_B(g)\\
&= (\bar{\alpha}_M(x))_B(\widetilde{f}_B(g))\\
&= (\bar{\alpha}_M(x))_B(f \circ g)\\
&= (\alpha_B)^*(x^\sharp \circ \boldsymbol{y}(f \circ g))\\
&= (\alpha_B)^*(x^\sharp \circ \boldsymbol{y}f \circ \boldsymbol{y}g))\\
&= (\alpha_B)^*(((Xf)(x))^\sharp \circ \boldsymbol{y}g)\\
&= (\bar{\alpha}_N((Xf)(x)))_B(g)
\end{aligned}
$$

Thus $\bar{\alpha}$ is a natural transformation.

Recall that $(\bar{\alpha})^{\boldsymbol{y}c} = \widehat{\bar{\alpha} \circ \mathrm{ev}_X}$, where $\mathrm{ev}_X : X^{\boldsymbol{y}c} \times \boldsymbol{y}c \to X$ is the application map for all objects $X \in \widehat{\mathbb{C}}$ and $\widehat{\gamma}$ denotes the unique curried map to an exponential object associated with γ. We must show that $\alpha = \epsilon_Y \circ (\bar{\alpha})^{\boldsymbol{y}c}$, or equivalently that the components at each object $A \in \mathbb{C}$ on both sides are equal for all elements $\beta \in X^{\boldsymbol{y}c}(A)$. Let $A \in \mathbb{C}$ and $\beta : \boldsymbol{y}A \times \boldsymbol{y}c \Rightarrow X$. Then

$$\begin{aligned}
&(\epsilon_Y)_A(((\bar{\alpha})^{\boldsymbol{y}c})_A(\beta)) \\
&= (\epsilon_Y)_A((\widehat{\bar{\alpha} \circ \mathrm{ev}_X})_A(\beta)) \\
&= (\epsilon_Y)_A((\bar{\alpha} \circ \mathrm{ev}_X) \circ (\widehat{\beta} \times \mathrm{id}_{\boldsymbol{y}c})) \\
&= (\epsilon_Y)_A(\bar{\alpha} \circ (\mathrm{ev}_X \circ (\widehat{\beta} \times \mathrm{id}_{\boldsymbol{y}c}))) \\
&= (\epsilon_Y)_A(\bar{\alpha} \circ \beta) \\
&= (((\bar{\alpha} \circ \beta)_{A\times c}(\pi_1, \pi_2))_A)(\mathrm{id}_{A\times c}) \\
&= (((\bar{\alpha}_{A\times c} \circ \beta_{A\times c})(\pi_1, \pi_2))_A)(\mathrm{id}_{A\times c}) \\
&= ((\bar{\alpha}_{A\times c}(\beta_{A\times c}(\pi_1, \pi_2)))_A)(\mathrm{id}_{A\times c}) \\
&= (\alpha_A)^*((\beta_{A\times c}(\pi_1, \pi_2))^\sharp \circ \boldsymbol{y}(\mathrm{id}_{A\times c})) \\
&= (\alpha_A)^*((\beta_{A\times c}(\pi_1, \pi_2))^\sharp) \\
&= \alpha_A(\beta)
\end{aligned}$$

Finally, we must show that $\bar{\alpha}$ is unique. ¡?out of time!¿ This is a consequence of the condition $((\bar{\alpha}_{A\times c}(\beta_{A\times c}(\pi_1, \pi_2)))_A)(\mathrm{id}_{A\times c})$, the fact that products are universal, and the condition that $\bar{\alpha}$ is natural.

Q4.2

We first show that $\boldsymbol{Sub}$ is cartesian closed, which means showing that it contains finite products and exponentials. To show that $\boldsymbol{Sub}$ contains finite products, it suffices to show that it contains a terminal object and binary products.

(terminal object) Consider the pair $(\mathbf{1}, \mathbf{1})$, where $\mathbf{1}$ is a terminal object in $\boldsymbol{Set}$. Since $\mathbf{1} \subseteq \mathbf{1}$, this pair is an object in $\boldsymbol{Sub}$. Let (X, S) be an object in $\boldsymbol{Sub}$. Consider the unique map ${}_X! : X \to \mathbf{1}$. Note that ${}_X!(x) \in \mathbf{1}$ for all $x \in X$, so in particular $x \in S \implies {}_X!(x) \in \mathbf{1}$, therefore ${}_X! : (X, S) \to (\mathbf{1}, \mathbf{1})$ is a morphism in $\boldsymbol{Sub}$. Furthermore, if $f : (X, S) \to (\mathbf{1}, \mathbf{1})$ is a morphism in $\boldsymbol{Sub}$ then $f : X \to \mathbf{1}$ is a morphism in $\boldsymbol{Set}$, and therefore $f = {}_X!$ since ${}_X!$ is unique. Thus $(\mathbf{1}, \mathbf{1})$ is terminal in $\boldsymbol{Sub}$ with ${}_{(X,S)}! = {}_X!$.

(binary products) Let (X, S) and (Y, T) be objects in $\boldsymbol{Sub}$. Consider the pair $(X \times Y, S \times T)$. Since $S \times T \subseteq X \times Y$, because $S \subseteq X$ and $T \subseteq Y$, this pair is an object in $\boldsymbol{Sub}$. Now if $(x, y) \in S \times T$ then $\pi_1(x, y) = x \in S$ and $\pi_2(x, y) = y \in T$, so $\pi_1 : (X \times Y, S \times T) \to (X, S)$ and $\pi_2 : (X \times Y, S \times T) \to (Y, T)$ are morphisms in $\boldsymbol{Sub}$. Furthermore, if (P, Q) is an object in $\boldsymbol{Sub}$ with projections $p_1 : (P, Q) \to (X, S)$ and $p_2 : (P, Q) \to (Y, T)$, then $p_1 : P \to X$ and $p_2 : P \to Y$ are morphisms in $\boldsymbol{Set}$ so we may consider $\langle p_1, p_2 \rangle : P \to X \times Y$, and if $x \in Q$ then $p_1(x) \in S$ and $p_2(x) \in T$, so $\langle p_1, p_2 \rangle(x) \in S \times T$, and hence $\langle p_1, p_2 \rangle : (P, Q) \to (X \times Y, S \times T)$ is a morphism in $\boldsymbol{Sub}$. That $\langle p_1, p_2 \rangle$ is the unique map satisfying $\pi_1 \circ \langle p_1, p_2 \rangle = p_1$ and $\pi_2 \circ \langle p_1, p_2 \rangle = p_2$ in $\boldsymbol{Sub}$ follows from the fact that this is true in $\boldsymbol{Set}$, that maps in $\boldsymbol{Sub}$ are all maps in $\boldsymbol{Set}$, and that composition in $\boldsymbol{Sub}$ is as in $\boldsymbol{Set}$. Therefore $(X \times Y, S \times T)$ is a product in $\boldsymbol{Sub}$ with projections π_1 and π_2.

(exponentials) Let (X, S) and (Y, T) be objects in $\boldsymbol{Sub}$. Consider the pair $(Y^X, Y^X_{(S,T)})$ where $Y^X_{(S,T)} = \{f \in Y^X \mid \forall x \in S.\ f(x) \in T\}$. Clearly $Y^X_{(S,T)} \subseteq Y^X$, so this pair is an object in $\boldsymbol{Sub}$. Consider the application map $\mathrm{ev} : Y^X \times X \to Y$ given by $\mathrm{ev}(f, x) = f(x)$. If $(f, x) \in Y^X_{(S,T)} \times S$ then $f(x) \in T$ because $x \in S$. Therefore $\mathrm{ev} : (Y^X, Y^X_{(S,T)}) \times (X, S) \to (Y, T)$ is a morphism in $\boldsymbol{Sub}$. Furthermore, if (E, F) is an object in $\boldsymbol{Sub}$ with a morphism $e : (E, F) \times (X, S) \to (Y, T)$, then $e : E \times X \to Y$ is a morphism in $\boldsymbol{Set}$ so there is a unique curried map $\hat{e} : E \to Y^X$ such that $\text{ev} \circ (\hat{e} \times \text{id}_X) = e$. Let $g \in F$. For all $x \in X$ we have $\hat{e}(g)(x) = \text{ev}(\hat{e}(g), \text{id}_X(x)) = e(g, x)$. Additionally, if $x \in S$ then $e(g, x) \in T$, since e is a morphism in $\boldsymbol{Sub}$, and hence $\hat{e}(g) \in Y^X_{(S,T)}$. It follows that $\hat{e} : (E, F) \to (Y^X, Y^X_{(S,T)})$ is a morphism in $\boldsymbol{Sub}$. That $\hat{e}$ is the unique morphism in $\boldsymbol{Sub}$ satisfying $\text{ev} \circ (\hat{e} \times \text{id}_X) = e$ follows from the fact that this is true in $\boldsymbol{Set}$, that maps in $\boldsymbol{Sub}$ are all maps in $\boldsymbol{Set}$, and that composition, identities, product morphisms, and projections in $\boldsymbol{Sub}$ are as in $\boldsymbol{Set}$. Therefore $(Y^X, Y^X_{(S,T)})$ with ev is an exponential in $\boldsymbol{Sub}$.

We now show that $\boldsymbol{Sub}$ is distributive, which means showing that it is cocartesian, that is, contains finite coproducts, and that certain canonical maps are isomorphisms. To show that $\boldsymbol{Sub}$ contains finite coproducts, it suffices to show that it contains an initial object and binary coproducts.

(initial object) Consider the pair $(\emptyset, \emptyset)$. Since $\emptyset \subseteq \emptyset$ this is an object in $\boldsymbol{Sub}$. Let (X, S) be an object in $\boldsymbol{Sub}$. The unique map $!_X : \emptyset \to X$ vacuously satisfies $x \in \emptyset \implies !_X(x) \in S$ for all $x \in \emptyset$, so it is also a morphism $(\emptyset, \emptyset) \to (X, S)$ in $\boldsymbol{Sub}$. Finally, any morphism $(\emptyset, \emptyset) \to (X, S)$ in $\boldsymbol{Sub}$ must be a morphism $\emptyset \to X$ in $\boldsymbol{Set}$, and hence must equal $!_X$, which is unique in $\boldsymbol{Set}$. So $!_X$ is the unique map to each (X, S), and $(\emptyset, \emptyset)$ is initial in $\boldsymbol{Sub}$.

(binary coproducts) Let (X, S) and (Y, T) be objects in $\boldsymbol{Sub}$. Consider the pair $(X{+}Y, S{+}T)$. Since[1] $S{+}T \subseteq X{+}Y$, this is an object in $\boldsymbol{Sub}$. Now if $x \in S$ then $\iota_1(x) \in S{+}T$, and if $y \in T$ then $\iota_2(y) \in S{+}T$, therefore $\iota_1 : (X, S) \to (X{+}Y, S{+}T)$ and $\iota_2 : (Y, T) \to (X{+}Y, S{+}T)$ are morphisms in $\boldsymbol{Sub}$. Furthermore, if (P, Q) is an object in $\boldsymbol{Sub}$ with injections $i_1 : (X, S) \to (P, Q)$ and $i_2 : (Y, T) \to (P, Q)$, then $i_1 : X \to P$ and $i_2 : Y \to P$ are morphisms in $\boldsymbol{Set}$ so we may consider $[i_1, i_2] : X + Y \to P$, and if $x \in S$ then $i_1(x) \in Q$ and if $y \in T$ then $i_2(x) \in Q$, so if $s \in S + T$ then $[i_1, i_2](s) \in Q$, and hence $[i_1, i_2] : (X + Y, S + T) \to (P, Q)$ is a morphism in $\boldsymbol{Sub}$. That $[i_1, i_2]$ is the unique map satisfying $[i_1, i_2] \circ \iota_1 = i_1$ and $[i_1, i_2] \circ \iota_2 = i_2$ in $\boldsymbol{Sub}$ follows from the fact that this is true in $\boldsymbol{Set}$, that maps in $\boldsymbol{Sub}$ are all maps in $\boldsymbol{Set}$, and that composition in $\boldsymbol{Sub}$ is as in $\boldsymbol{Set}$. Therefore $(X + Y, S + T)$ is a coproduct in $\boldsymbol{Sub}$ with injections ι_1 and ι_2.

(distributivity with initial object) We must show that the canonical morphism $!_{X \times \emptyset} : (\emptyset, \emptyset) \to (X, S) \times (\emptyset, \emptyset)$ is an isomorphism for all objects (X, S) in $\boldsymbol{Sub}$. But $!_{X \times \emptyset}$ is an identity morphism in $\boldsymbol{Set}$ for the object $X \times \emptyset$, which is equal to $\emptyset$. Identities in $\boldsymbol{Sub}$ are as in $\boldsymbol{Set}$, so $!_{X \times \emptyset}$ is the identity for $(X, S) \times (\emptyset, \emptyset)$, and hence is an isomorphism with itself as inverse.

(distributivity with binary coproducts) We must show that the canonical morphism

$$[\text{id}_{(X,S)} \times \iota_1, \text{id}_{(X,S)} \times \iota_2] : (X, S) \times (Y, T) + (X, S) \times (Z, U) \to (X, S) \times ((Y, T) + (Z, U))$$

is an isomorphism for all objects (X, S), (Y, T), and (Z, U) in $\boldsymbol{Sub}$. Since identities, composition, morphism products, morphism coproducts, projections, and injections in $\boldsymbol{Sub}$ are all as in $\boldsymbol{Set}$, the corresponding map in $\boldsymbol{Set}$, namely $[\text{id}_X \times \iota_1, \text{id}_X \times \iota_2] : X \times Y + X \times Z \to X \times (Y + Z)$, is canonical and has an inverse since $\boldsymbol{Set}$ is distributive. Let $(x, w) \in S \times (T{+}U)$. Either $w = (y, 0)$ for some $y \in T$ or $w = (z, 1)$ for some $z \in U$. In the first case, $[\text{id}_X \times \iota_1, \text{id}_X \times \iota_2]^{-1}(x, w) = ((x, y), 0)$, and in the second case $[\text{id}_X \times \iota_2, \text{id}_X \times \iota_2]^{-1}(x, w) = ((x, z), 1)$. In both cases the result is in $S \times T + S \times U$. Therefore the inverse $[\text{id}_X \times \iota_1, \text{id}_X \times \iota_2]^{-1} : (X, S) \times ((Y, T) + (Z, U)) \to (X, S) \times (Y, T) + (X, S) \times (Z, U)$ is a morphism in $\boldsymbol{Sub}$. Since composition and identities, hence isomorphisms where they exist, are as in $\boldsymbol{Set}$, it follows that the canonical morphism and its inverse are isomorphic in $\boldsymbol{Sub}$ as required.

[1] If $x \in S + T$ then either $x = (x_0, 0)$ for some $x_0 \in S$, hence $x \in X + Y$ since $x_0 \in X$, or $x = (y_0, 1)$ for some $y_0 \in T$, hence $x \in X + Y$ since $y_0 \in Y$