L12 Mid-Term Test

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1. On Galois connections

(a) $\Longrightarrow$ In fact, g is monotone no matter whether f is monotone.

$$\boxed{\forall x \in Q, y \in Q.\ x \leq_{\mathsf{Q}} y \implies g(x) \leq_{\mathsf{P}} g(y)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{g(x) \leq_{\mathsf{P}} g(x)}\ \text{reflexivity}}{f(g(x)) \leq_{\mathsf{Q}} x}\ (1) \qquad \dfrac{}{x \leq_{\mathsf{Q}} y}\ \text{assumption}}{f(g(x)) \leq_{\mathsf{Q}} y}\ \text{transitivity}}{g(x) \leq_{\mathsf{P}} g(y)}\ (1)$$

$\Longleftarrow$ Symmetrically, f is monotone. In the main theorem, instead of "iff" we have "and".

$$\boxed{\forall x \in P, y \in P.\ x \leq_{\mathsf{P}} y \implies f(x) \leq_{\mathsf{Q}} f(y)}$$

$$\dfrac{\dfrac{\dfrac{}{x \leq_{\mathsf{P}} y}\ \text{assumption} \qquad \dfrac{\dfrac{}{f(y) \leq_{\mathsf{Q}} f(y)}\ \text{reflexivity}}{y \leq_{\mathsf{P}} g(f(y))}\ (1)}{x \leq_{\mathsf{P}} g(f(y))}\ \text{transitivity}}{f(x) \leq_{\mathsf{Q}} f(y)}\ (1)$$

(b) **(1)**

$$\boxed{\forall S \subseteq Y, T \subseteq Y.\ S \subseteq T \implies f^{-1}[S] \subseteq f^{-1}[T]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{S \subseteq T}\ \text{assumption}}{\forall x \in X.\ f(x) \in S \implies f(x) \in T}\ \text{consequence of} \subseteq}{\forall x \in X.\ x \in f^{-1}[S] \implies x \in f^{-1}[T]}\ \text{definition of } f^{-1}}{f^{-1}[S] \subseteq f^{-1}[T]}\ \text{definition of} \subseteq$$

(2) Define $\exists_f$ by $\exists_f(S) = \{f(x) \mid x \in S\}$. To show $\exists_f \dashv f^{-1}[\_]$ is a Galois connection is to show

$$\boxed{\forall S \subseteq X, T \subseteq Y.\ \exists_f(S) \subseteq T \iff S \subseteq f^{-1}[T]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\forall x \in S.\ f(x) \in T) \iff (\forall x \in S.\ f(x) \in T)}}{(\forall y.\ (\exists x \in S.\ y = f(x)) \implies y \in T) \iff (\forall x.\ x \in S \implies f(x) \in T)}\ \text{simplification}}{\{f(x) \mid x \in S\} \subseteq T \iff S \subseteq \{x \mid f(x) \in T\}}\ \text{definition of} \subseteq}{\exists_f(S) \subseteq T \iff S \subseteq f^{-1}[T]}\ \text{definitions of } \exists_f \text{ and } f^{-1}$$

(3) Define $\forall_f$ by $\forall_f(T) = \{y \in Y \mid \forall x \in X.\, y = f(x) \implies x \in T\}$. To show $f^{-1}[\_] \dashv \forall_f$ is a Galois connection is to show

$$\boxed{\forall S \subseteq Y, T \subseteq X.\; f^{-1}[S] \subseteq T \iff S \subseteq \forall_f(T)}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{(\forall x.\, f(x) \in S \implies x \in T) \iff (\forall x.\, f(x) \in S \implies x \in T)}}{(\forall x.\, f(x) \in S \implies x \in T) \iff (\forall y.\, y \in S \implies \forall x.\, y = f(x) \implies x \in T)}\;\text{simplification}}{\{x \mid f(x) \in S\} \subseteq T \iff S \subseteq \{y \mid \forall x.\, y = f(x) \implies x \in T\}}\;\text{definition of } \subseteq}{f^{-1}[S] \subseteq T \iff S \subseteq \forall_f(T)}\;\text{definitions of } f^{-1} \text{ and } \forall_f$$

2. On distributive categories

(a) Terminal object Let $1 \in \mathcal{S}$ be terminal, and let ${}_{S\times 1}! : S \times 1 \to 1$ be the unique morphism from $S \times 1$. We show that the S-action $(1, {}_{S\times 1}!)$ is terminal in S-act. Let $(A, \alpha) \in S$-act. Then ${}_A! : A \to 1$ in $\mathcal{S}$ gives a morphism $(A, \alpha) \to (1, {}_{S\times 1}!)$ in S-act:

$$\begin{array}{ccc} S \times A & \xrightarrow{\mathrm{id}_S \times {}_A!} & S \times 1 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle {}_{S\times 1}!} \\ A & \xrightarrow[{}_A!]{} & 1 \end{array}$$

There is only one morphism $S \times A \to 1$ in $\mathcal{S}$, therefore ${}_A! \circ \alpha = {}_{S\times 1}! \circ (\mathrm{id}_S \times {}_A!)$ as required. Uniqueness of ${}_A!$ in S-act follows from its uniqueness in $\mathcal{S}$: the only candidates for $(A, \alpha) \to (1, {}_{S\times 1}!)$ are morphisms $A \to 1$ in $\mathcal{S}$. Thus $(1, {}_{S\times 1}!)$ is terminal.

Binary products Let (A, α) and (B, β) be objects in S-act. The object

$$(A, \alpha) \times (B, \beta) = (A \times B, \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle)$$

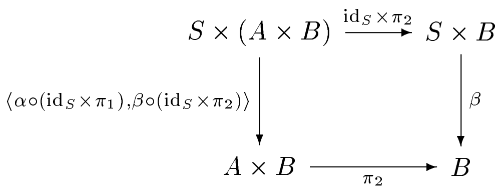
is their product, with projections π_1 and π_2. To see that this is an object of S-act, observe that $A \times B$ is an object of $\mathcal{S}$ since $\mathcal{S}$ contains products, and check the types of the morphisms:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\mathrm{id}_S : S \to S} \quad \cfrac{}{\pi_1 : A \times B \to A}}{\mathrm{id}_S \times \pi_1 : S \times (A \times B) \to S \times A} \quad \cfrac{}{\alpha : S \times A \to A}}{\alpha \circ (\mathrm{id}_S \times \pi_1) : S \times (A \times B) \to A} \quad \cfrac{\cfrac{\cfrac{}{\mathrm{id}_S : S \to S} \quad \cfrac{}{\pi_2 : A \times B \to B}}{\mathrm{id}_S \times \pi_2 : S \times (A \times B) \to S \times B} \quad \cfrac{}{\beta : S \times B \to B}}{\beta \circ (\mathrm{id}_S \times \pi_2) : S \times (A \times B) \to B}}{\langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle : S \times (A \times B) \to A \times B}$$

We check that π_1 and π_2 are morphisms in S-act.

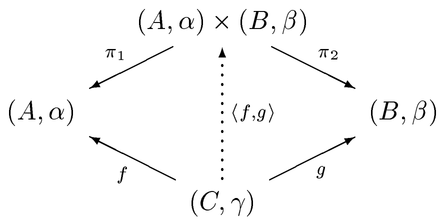
$$\begin{array}{ccc} S \times (A \times B) & \xrightarrow{\mathrm{id}_S \times \pi_1} & S \times A \\ {\scriptstyle \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A \times B & \xrightarrow[\pi_1]{} & A \end{array}$$

$$\frac{\overline{\alpha \circ (\mathrm{id}_S \times \pi_1) = \alpha \circ (\mathrm{id}_S \times \pi_1)}}{\pi_1 \circ \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle = \alpha \circ (\mathrm{id}_S \times \pi_1)} \ \text{universal property of } A \times B$$

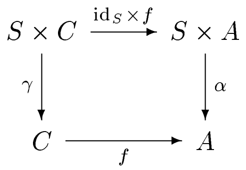


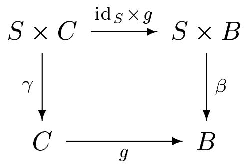
$$\frac{\overline{\beta \circ (\mathrm{id}_S \times \pi_2) = \beta \circ (\mathrm{id}_S \times \pi_2)}}{\pi_2 \circ \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle = \beta \circ (\mathrm{id}_S \times \pi_2)} \ \text{universal property of } A \times B$$

Finally, we check that $(A, \alpha) \times (B, \beta)$ satisfies the universal property for products.

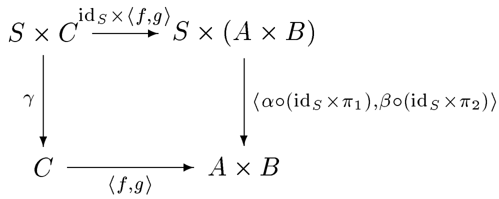


Suppose (C, γ) is another object with projections $f : (C, \gamma) \to (A, \alpha)$ and $g : (C, \gamma) \to (B, \beta)$. Since f and g are morphisms in S-act, the following diagrams commute.





We can take the product of morphisms $\langle f, g \rangle : (C, \gamma) \to (A, \alpha) \times (B, \beta)$ in S-act to be the product $\langle f, g \rangle : C \to A \times B$ in $\mathcal{S}$. This is a morphism in S-act:



$$\begin{aligned}
\langle f, g \rangle \circ \gamma &= \langle f \circ \gamma, g \circ \gamma \rangle && \text{composition with product} \\
&= \langle \alpha \circ (\mathrm{id}_S \times f), \beta \circ (\mathrm{id}_S \times g) \rangle && f, g \text{ are morphisms in } S\text{-act} \\
&= \langle \alpha \circ (\mathrm{id}_S \times (\pi_1 \circ \langle f, g \rangle)), \beta \circ (\mathrm{id}_S \times (\pi_2 \circ \langle f, g \rangle)) \rangle && \text{universal property of } A \times B \\
&= \langle \alpha \circ ((\mathrm{id}_S \circ \mathrm{id}_S) \times (\pi_1 \circ \langle f, g \rangle)), \beta \circ ((\mathrm{id}_S \circ \mathrm{id}_S) \times (\pi_2 \circ \langle f, g \rangle)) \rangle && \mathrm{id}_S \text{ is identity} \\
&= \langle \alpha \circ (\mathrm{id}_S \times \pi_1) \circ (\mathrm{id}_S \times \langle f, g \rangle), \beta \circ (\mathrm{id}_S \times \pi_2) \circ (\mathrm{id}_S \times \langle f, g \rangle) \rangle && \text{composition of direct products} \\
&= \langle \alpha \circ (\mathrm{id}_S \times \pi_1), \beta \circ (\mathrm{id}_S \times \pi_2) \rangle \circ (\mathrm{id}_S \times \langle f, g \rangle) && \text{composition with product}
\end{aligned}$$

Finally, the universal property diagram commutes because it commutes in $\mathcal{S}$: we only need consider the first component of each object since morphisms in S-act are morphisms in $\mathcal{S}$ on the first components, and composition in S-act is composition in $\mathcal{S}$. Similarly, the product of morphisms $\langle f, g\rangle$ is unique in S-act because it is unique in $\mathcal{S}$.

We have shown that S-act has a terminal object, and has a binary product for each pair of objects. It follows that S-act is cartesian.

(b) Suppose $\mathcal{S}$ is distributive. Then for any objects A, B, and C, the canonical map $[\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]$ is an isomorphism $(A\times B) + (A \times C) \simeq A \times (B + C)$, and for any object A, the canonical map $\langle !_A, \mathrm{id}_0\rangle$ is an isomorphism $0 \simeq A \times 0$. To show that S-act is distributive is to show that these maps exist and are isomorphisms in S-act. For the maps to exist, we must first show that S-act is bicartesian. We have seen that S-act is cartesian when $\mathcal{S}$ is cartesian. We proceed to show that S-act is cocartesian when $\mathcal{S}$ is distributive, and then that S-act is also distributive.

Initial object Let $0 \in \mathcal{S}$ be initial, and let $!_{S\times 0} : 0 \to S \times 0$ be the unique morphism to $S \times 0$. Since $\mathcal{S}$ is distributive, the map $!_{S\times 0}$ is an isomorphism and has an inverse. We show that the S-action $(0, (!_{S\times 0})^{-1})$ is initial in S-act. Let $(A, \alpha) \in S$-act. Then $!_A : 0 \to A$ in $\mathcal{S}$ gives a morphism $(0, (!_{S\times 0})^{-1}) \to (A, \alpha)$ in S-act:

$$\begin{array}{ccc} S \times 0 & \xrightarrow{\mathrm{id}_S \times !_A} & S \times A \\ {\scriptstyle (!_{S\times 0})^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ 0 & \xrightarrow[!_A]{} & A \end{array}$$

There is only one morphism $0 \to A$ in $\mathcal{S}$, therefore $\alpha \circ (\mathrm{id}_S \times !_A) \circ !_{S\times 0} = !_A$. Composing on the right with the inverse isomorphism, we obtain $\alpha \circ (\mathrm{id}_S \times !_A) = !_A \circ (!_{S\times 0})^{-1}$ as required. Uniqueness of $!_A$ in S-act follows from its uniqueness in $\mathcal{S}$. Thus $(0, (!_{S\times 0})^{-1})$ is initial.

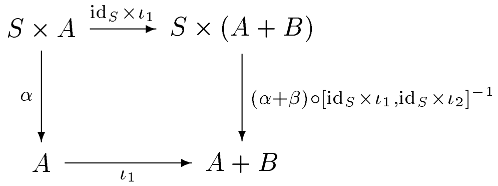
Binary coproducts Let (A, α) and (B, β) be objects in S-act. The object

$$(A, \alpha) + (B, \beta) = (A + B, (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1})$$

is their sum, with injections ι_1 and ι_2. To see that this is an object of S-act, observe that $A + B$ is an object of $\mathcal{S}$ since $\mathcal{S}$ contains coproducts, and check the types of the morphisms:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathrm{id}_S : S \to S} \quad \overline{\iota_1 : A \to A + B}}{\mathrm{id}_S \times \iota_1 : S \times A \to S \times (A + B)} \quad \dfrac{\overline{\mathrm{id}_S : S \to S} \quad \overline{\iota_2 : B \to A + B}}{\mathrm{id}_S \times \iota_2 : S \times B \to S \times (A + B)}}{[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] : (S \times A) + (S \times B) \to S \times (A + B)}}{[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} : S \times (A + B) \to (S \times A) + (S \times B)} \quad \dfrac{\overline{\alpha : S \times A \to A} \quad \overline{\beta : S \times B \to B}}{(\alpha + \beta) : (S \times A) + (S \times B) \to A + B}}{(\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} : S \times (A + B) \to A + B}$$

We check that ι_1 and ι_2 are morphisms in S-act.

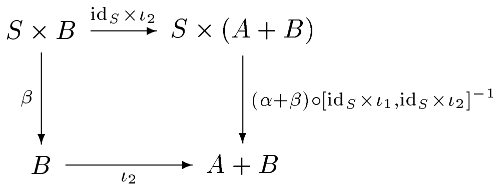


$$\frac{}{\iota_1 \circ \alpha = (\alpha + \beta) \circ \iota_1}\ \text{universal property of } (S \times A) + (S \times B)^1$$

$$\frac{\iota_1 \circ \alpha = (\alpha + \beta) \circ \iota_1}{\iota_1 \circ \alpha = (\alpha + \beta) \circ \mathrm{id}_{(S\times A)+(S\times B)} \circ \iota_1}\ \mathrm{id}_{(S\times A)+(S\times B)} \text{ is identity}$$

$$\frac{\iota_1 \circ \alpha = (\alpha + \beta) \circ \mathrm{id}_{(S\times A)+(S\times B)} \circ \iota_1}{\iota_1 \circ \alpha = (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] \circ \iota_1}\ \text{composition of inverses}$$

$$\frac{\iota_1 \circ \alpha = (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] \circ \iota_1}{\iota_1 \circ \alpha = (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \circ (\mathrm{id}_S \times \iota_1)}\ \text{universal property of } (S \times A) + (S \times B)$$



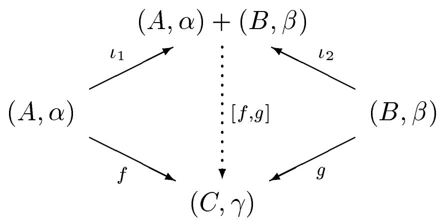
$$\frac{}{\iota_2 \circ \beta = (\alpha + \beta) \circ \iota_2}\ \text{universal property of } (S \times A) + (S \times B)$$

$$\frac{\iota_2 \circ \beta = (\alpha + \beta) \circ \iota_2}{\iota_2 \circ \beta = (\alpha + \beta) \circ \mathrm{id}_{(S\times A)+(S\times B)} \circ \iota_2}\ \mathrm{id}_{(S\times A)+(S\times B)} \text{ is identity}$$

$$\frac{\iota_2 \circ \beta = (\alpha + \beta) \circ \mathrm{id}_{(S\times A)+(S\times B)} \circ \iota_2}{\iota_2 \circ \beta = (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] \circ \iota_2}\ \text{composition of inverses}$$

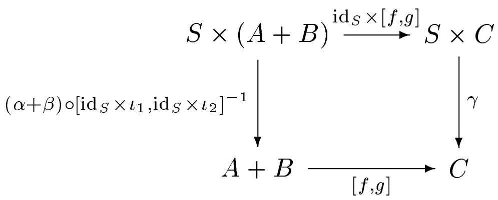
$$\frac{\iota_2 \circ \beta = (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] \circ \iota_2}{\iota_2 \circ \beta = (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \circ (\mathrm{id}_S \times \iota_2)}\ \text{universal property of } (S \times A) + (S \times B)$$

Finally, we check that $(A, \alpha) + (B, \beta)$ satisfies the universal property for coproducts.



Suppose (C, γ) is another object with injections $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (C, \gamma)$. Since f and g are morphisms in S-act, the following diagrams commute.

We can take the sum of morphisms $[f, g] : (A, \alpha) + (B, \beta) \to (C, \gamma)$ in S-act to be the sum $[f, g] : A + B \to C$ in S.



First, observe:

$$
\begin{aligned}
&[\mathrm{id}_S \times f, \mathrm{id}_S \times g] \\
&= [\mathrm{id}_S \times ([f, g] \circ \iota_1), \mathrm{id}_S \times ([f, g] \circ \iota_2)] && \text{universal property of } A + B \\
&= [(\mathrm{id}_S \circ \mathrm{id}_S) \times ([f, g] \circ \iota_1), (\mathrm{id}_S \circ \mathrm{id}_S) \times ([f, g] \circ \iota_2)] && \mathrm{id}_S \text{ is identity} \\
&= [(\mathrm{id}_S \times [f, g]) \circ (\mathrm{id}_S \times \iota_1), (\mathrm{id}_S \times [f, g]) \circ (\mathrm{id}_S \times \iota_2)] && \text{composition of direct products} \\
&= (\mathrm{id}_S \times [f, g]) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2] && \text{composition with sum}
\end{aligned}
$$

Composing on the left with γ and on the right with $[\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1}$, we get

$$\gamma \circ [\mathrm{id}_S \times f, \mathrm{id}_S \times g] \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} = \gamma \circ (\mathrm{id}_S \times [f, g])$$

Hence,

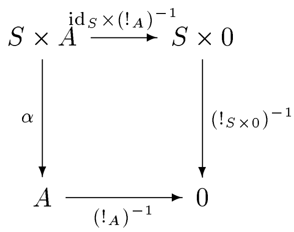
$$
\begin{aligned}
&\gamma \circ (\mathrm{id}_S \times [f, g]) \\
&= \gamma \circ [\mathrm{id}_S \times f, \mathrm{id}_S \times g] \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} \\
&= [\gamma \circ (\mathrm{id}_S \times f), \gamma \circ (\mathrm{id}_S \times g)] \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} && \text{composition with sum} \\
&= [f \circ \alpha, g \circ \beta] \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} && f, g \text{ morphisms in } S\text{-}\mathsf{act} \\
&= [f, g] \circ (\alpha + \beta) \circ [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]^{-1} && \text{composition of sums}
\end{aligned}
$$

This equality represents commutativity of the diagram above, and thus implies $[f, g]$ is a morphism in S-act.

By the same reasoning we used for products, the universal property diagram for coproducts commutes in S-act because it commutes in $\mathcal{S}$, and the sum of morphisms $[f, g]$ is unique in S-act because it is unique in $\mathcal{S}$.

Distributivity

With initial object Let (A, α) be an object, and let $(0, (!_{S\times 0})^{-1})$ be an initial object in S-act. Since $\mathcal{S}$ is distributive, the unique map $!_A : 0 \to A$ is an isomorphism. We have already seen that $!_A$ is a morphism in S-act, indeed the unique morphism $(0, (!_{S\times 0})^{-1}) \to (A, \alpha)$. We now show $(!_A)^{-1}$ is also a morphism in S-act. It will follow that $!_A$ is an isomorphism in S-act, since identities and composition are inherited from $\mathcal{S}$.



$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \circ (\mathrm{id}_S \times !_A) = !_A \circ (!_{S\times 0})^{-1}}\ !_A \text{ is a morphism in } S\text{-}\mathsf{act}}{(!_A)^{-1} \circ \alpha \circ (\mathrm{id}_S \times !_A) \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big) = (!_A)^{-1} \circ !_A \circ (!_{S\times 0})^{-1} \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big)}}{(!_A)^{-1} \circ \alpha \circ (\mathrm{id}_S \times !_A) \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big) = \mathrm{id}_0 \circ (!_{S\times 0})^{-1} \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big)}\ \text{inverses}}{(!_A)^{-1} \circ \alpha \circ \big((\mathrm{id}_S \circ \mathrm{id}_S) \times (!_A \circ (!_A)^{-1})\big) = \mathrm{id}_0 \circ (!_{S\times 0})^{-1} \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big)}\ \text{composition of products}}{(!_A)^{-1} \circ \alpha \circ (\mathrm{id}_S \times \mathrm{id}_A) = (!_{S\times 0})^{-1} \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big)}\ \text{identities and inverses}}{(!_A)^{-1} \circ \alpha \circ \mathrm{id}_{S\times A} = (!_{S\times 0})^{-1} \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big)}\ \text{product of identities}}{(!_A)^{-1} \circ \alpha = (!_{S\times 0})^{-1} \circ \big(\mathrm{id}_S \times (!_A)^{-1}\big)}\ \mathrm{id}_{S\times A} \text{ is identity}
$$

With binary coproducts Let (A, α), (B, β), and (C, γ) be objects in S-act. Since $\mathcal{S}$ is distributive, the canonical map $[\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2] : (A \times B) + (A \times C) \to A \times (B + C)$ is an isomorphism. We now show that it and its inverse are both morphisms in S-act. That they are isomorphisms in S-act will follow immediately, since identities and composition are as in $\mathcal{S}$.

We first calculate the objects between which we want $[\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]$ to be a morphism in S-act. To simplify the notation, we introduce the following (polymorphic) abbreviations.

$$
\begin{aligned}
p_1 &= \mathrm{id}_S \times \pi_1 \\
p_2 &= \mathrm{id}_S \times \pi_2 \\
s &= [\mathrm{id}_S \times \iota_1, \mathrm{id}_S \times \iota_2]
\end{aligned}
$$

Also, we denote composition $\_ \circ \_$ by juxtaposition $\_\_$.

$$
\dfrac{\dfrac{\overline{(A,\alpha)} \quad \overline{(B,\beta)}}{(A \times B, \langle \alpha p_1, \beta p_2 \rangle)} \quad \dfrac{\overline{(A,\alpha)} \quad \overline{(C,\gamma)}}{(A \times C, \langle \alpha p_1, \gamma p_2 \rangle)}}{\big((A \times B) + (A \times C), (\langle \alpha p_1, \beta p_2 \rangle + \langle \alpha p_1, \gamma p_2 \rangle) s^{-1}\big)}
$$

$$
\dfrac{\overline{(A,\alpha)} \quad \dfrac{\overline{(B,\beta)} \quad \overline{(C,\gamma)}}{\big(B + C, (\beta + \gamma) s^{-1}\big)}}{\big(A \times (B + C), \langle \alpha p_1, (\beta + \gamma) s^{-1} p_2 \rangle\big)}
$$

We now show $a = [\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]$ is a morphism in S-act.

$$
\begin{array}{ccc}
S \times ((A \times B) + (A \times C)) & \xrightarrow{\mathrm{id}_S \times a} & S \times (A \times (B + C)) \\
{\scriptstyle (\langle \alpha p_1, \beta p_2 \rangle + \langle \alpha p_1, \gamma p_2 \rangle) s^{-1}} \downarrow & & \downarrow {\scriptstyle \langle \alpha p_1, (\beta + \gamma) s^{-1} p_2 \rangle} \\
(A \times B) + (A \times C) & \xrightarrow[a]{} & A \times (B + C)
\end{array}
$$

$$
\begin{aligned}
&a\left(\langle \alpha p_1, \beta p_2\rangle + \langle \alpha p_1, \gamma p_2\rangle\right) s^{-1}\\
&= [\mathrm{id}_A \times \iota_1, \mathrm{id}_A \times \iota_2]\left(\langle \alpha p_1, \beta p_2\rangle + \langle \alpha p_1, \gamma p_2\rangle\right) s^{-1}\\
&= [(\mathrm{id}_A \times \iota_1)\langle \alpha p_1, \beta p_2\rangle, (\mathrm{id}_A \times \iota_2)\langle \alpha p_1, \gamma p_2\rangle] s^{-1}\\
&= [\langle \mathrm{id}_A \alpha p_1, \iota_1 \beta p_2\rangle, \langle \mathrm{id}_A \alpha p_1, \iota_2 \gamma p_2\rangle] s^{-1}\\
&= [\langle \alpha p_1, \iota_1 \beta p_2\rangle, \langle \alpha p_1, \iota_2 \gamma p_2\rangle] s^{-1}\\
&= [(\alpha \times \iota_1\beta)\langle p_1, p_2\rangle, (\alpha \times \iota_2\gamma)\langle p_1, p_2\rangle] s^{-1}\\
&= [\alpha \times \iota_1\beta, \alpha \times \iota_2\gamma](\langle p_1, p_2\rangle + \langle p_1, p_2\rangle) s^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\langle \alpha p_1, (\beta+\gamma)s^{-1}p_2\rangle(\mathrm{id}_S \times a)\\
&= (\alpha \times (\beta+\gamma)s^{-1})\langle p_1, p_2\rangle(\mathrm{id}_S \times a)
\end{aligned}
$$

3. On sections and regular, strong, and extremal monomorphisms

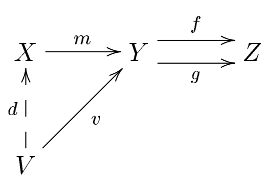
(a) section $\implies$ regular Let $m : X \to Y$ be a section of $s : Y \to X$, which means $s \circ m = \mathrm{id}_X$. (It follows, and in any case we assume, that m is a monomorphism.) Then m is an equaliser of id_Y and $m \circ s$, since

$$
\frac{\dfrac{\dfrac{}{m = m}}{m = m \circ \mathrm{id}_X}\ \mathrm{id}_X \text{ is identity}}{\mathrm{id}_Y \circ m = m \circ s \circ m}\ \mathrm{id}_Y \text{ is identity, } s \circ m = \mathrm{id}_X
$$

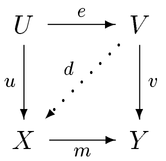
and m is universal: Specifically, given any $m' : Z \to Y$ satisfying $\mathrm{id}_Y \circ m' = m \circ s \circ m'$, we immediately have $m' = m \circ u$, where $u = s \circ m'$, since id_Y is the identity. This u is unique: If u' also satisfies $m' = m \circ u'$, then

$$
\frac{\dfrac{\dfrac{\dfrac{}{m \circ u' = m'}\ \text{assumption on } u'}{m \circ u' = m \circ s \circ m'}\ \text{asssumption on } m'}{u' = s \circ m'}\ m \text{ is mono}}{u' = u}\ \text{definition of } u
$$

regular $\implies$ strong Let $m : X \to Y$ be a monomorphism, so $\forall u, x : U \to X.\ m \circ u = m \circ x \implies u = x$. Furthermore, let m be regular, so there exist $f, g : Y \to Z$ such that m satisfies the following universal property: $\forall v : V \to Y.\ f \circ v = g \circ v \implies \exists! d : V \to X.\ v = m \circ d$.



Now let $e : U \to V$ be an epimorphism, so $\forall a, b : V \to Z.\ a \circ e = b \circ e \implies a = b$. Also let u and v morphisms such that the square below, without the diagonal, commutes.



We first show that there is a unique morphism $d : V \to X$ that makes the lower triangle commute.

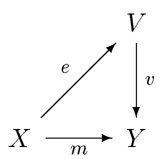
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f \circ m = g \circ m}\ m \text{ equalises } f, g}{f \circ m \circ u = g \circ m \circ u} \qquad \dfrac{}{m \circ u = v \circ e}\ \text{square commutes}}{f \circ v \circ e = g \circ v \circ e}}{f \circ v = g \circ v}\ e \text{ is epi}}{\exists! d.\ v = m \circ d}\ m \text{ equalises } f, g$$

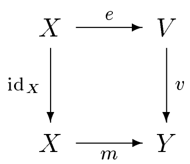
We now show the upper triangle also commutes, given this morphism d.

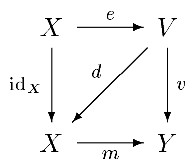
$$\dfrac{\dfrac{\dfrac{}{m \circ u = v \circ e}\ \text{square commutes} \qquad \dfrac{}{v = m \circ d}\ \text{lower triangle commutes}}{m \circ u = m \circ d \circ e}}{u = d \circ e}\ m \text{ is mono}$$

It follows that d is unique among morphisms that make both triangles commute. Therefore m is strong.

strong $\implies$ extremal Let $m : X \to Y$ be a strong monomorphism. Now let $e : X \to V$ be an epimorphism such that triangle on the left commutes.







From the commutativity of the triangle follows the commutativity of the square in the middle, because $m \circ \mathrm{id}_X = m$. Then, since m is strong, there exists a unique $d : V \to X$ such that both triangles on the right commute.

We get $e \circ d = \mathrm{id}_X$ from the commutativity of the upper triangle. Additionally,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v \circ e = m}\ \text{square commutes} \qquad \dfrac{}{v = m \circ d}\ \text{lower triangle commutes}}{m \circ d \circ e = m}}{m \circ d \circ e = m \circ \mathrm{id}_V}\ \mathrm{id}_V \text{ is identity}}{d \circ e = \mathrm{id}_V}\ m \text{ is mono}$$

Therefore e is an isomorphism, with inverse d.

4. On retractions, sections, pushouts, and coequalisers

(a) $\Longrightarrow$ Suppose the square below is a pushout.

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ B & \xrightarrow[p]{} & P \end{array}$$

In other words, $p \circ r = q \circ g$, and for any other arrows $p' : B \to P'$ and $q' : C \to P'$ such that $p' \circ r = q' \circ g$, there is a unique arrow $u : P \to P'$ such that $p' = u \circ p$ and $q' = u \circ q$.
Then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \circ r = q \circ g}\ \text{square commutes}}{p \circ r \circ s = q \circ g \circ s}}{p \circ \mathrm{id}_B = q \circ g \circ s}\ r \text{ retracts } s}{p = q \circ g \circ s}\ \mathrm{id}_B \text{ is identity}$$

Also, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \circ r = p \circ r}}{p \circ r = p \circ \mathrm{id}_B \circ r}\ \mathrm{id}_B \text{ is identity}}{p \circ r = p \circ r \circ s \circ r}\ r \text{ retracts } s \qquad \dfrac{}{p \circ r = q \circ g}\ \text{square commutes}}{q \circ g = q \circ g \circ s \circ r}$$

and, for any other $q' : C \to P'$ such that $q' \circ g = q' \circ g \circ s \circ r$, we can take $p' = q' \circ g \circ s$ and instantiate the universal property of the pushout to obtain a unique $u : P \to P'$ such that $p' = u \circ p$ and $q' = u \circ q$.

$$\begin{array}{ccccc} & & & & P' \\ & & & \overset{q'}{\nearrow} & \uparrow{\scriptstyle u} \\ A & \underset{g \circ s \circ r}{\overset{g}{\rightrightarrows}} & C & \xrightarrow[q]{} & P \end{array}$$

In fact, u is unique among arrows u' that make $q' = u' \circ q$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{u' \circ q \circ g \circ s = u' \circ q \circ g \circ s}}{q' \circ g \circ s = u' \circ q \circ g \circ s}\ q' = u' \circ q}{q' \circ g \circ s = u' \circ p}\ p = q \circ g \circ s}{p' = u' \circ p}\ p' = q' \circ g \circ s \qquad \dfrac{}{q' = u' \circ q}\ \text{assumption}}{u' = u}\ u \text{ is unique}$$

Therefore q coequalises g and $g \circ s \circ r$.

$\Longleftarrow$ Suppose now that $p = q \circ g \circ s$ and

$$A \underset{g \circ s \circ r}{\overset{g}{\rightrightarrows}} C \xrightarrow[q]{} P$$

is a coequaliser, which means $q \circ g = q \circ g \circ s \circ r$, and $\forall q' : C \to P'.\ q' \circ g = q' \circ g \circ s \circ r \implies \exists! u : P \to P'.\ q' = u \circ q$.
Then

$$\dfrac{\dfrac{\dfrac{}{q \circ g \circ s \circ r = q \circ g \circ s \circ r}}{q \circ g \circ s \circ r = q \circ g}\; q \circ g = q \circ g \circ s \circ r}{p \circ r = q \circ g}\; p = q \circ g \circ s$$

Also, if $p' : B \to P'$ and $q' : C \to P'$ are such that $p' \circ r = q' \circ g$, then we can instantiate the universal property of the coequaliser with q', because:

$$\dfrac{\dfrac{}{q' \circ g = p' \circ r}\;\text{assumption} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p' \circ r = q' \circ g}\;\text{assumption}}{p' \circ r \circ s = q' \circ g \circ s}}{p' \circ \mathrm{id}_B = q' \circ g \circ s}\; r \text{ retracts } s}{p' = q' \circ g \circ s}\; \mathrm{id}_B \text{ is identity}}{q' \circ g = q' \circ g \circ s \circ r}$$

Therefore there exists a unique u such that $q' = u \circ q$. In fact, we also have $p' = u \circ p$:

$$\dfrac{\dfrac{\dfrac{}{p' = q' \circ g \circ s}\;\text{as above}}{p' = u \circ q \circ g \circ s}\; q' = u \circ q}{p' = u \circ p}\; p = q \circ g \circ s$$

Every u' that satisfies $p' = u' \circ p$ and $q' = u' \circ q$ in particular satisfies $q' = u' \circ q$, and u is unique among the latter, so it is unique among the former. Therefore p and q are a pushout of r and g.