L11 Problem Set 2

Ramana Kumar (rk436)

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1 $\mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge \mathrm{LK}(T) \implies \mathrm{LD}(S \vec{\times} T)$

Let $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T$. As in the case proved in the lecture slides, the terms to be equated are t_{rhs} and $t_1 \otimes_T t_{\mathrm{lhs}}$, where

$$(\cdot, t_{\mathrm{lhs}}) = (s_2, t_2) \vec{\oplus} (s_3, t_3), \text{ and}$$
$$(\cdot, t_{\mathrm{rhs}}) = (s_1 \otimes_S s_2, t_1 \otimes_T t_2) \vec{\oplus} (s_1 \otimes_S s_3, t_1 \otimes_T t_3)$$

There are four possible values for t_{lhs}, depending on the value of $(s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3)$. They are

- $(t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)$, which equals $t_1 \otimes_T (t_2 \oplus_T t_3)$ by $\mathrm{LD}(T)$;
- $t_1 \otimes_T t_2$;
- $t_1 \otimes_T t_3$; or,
- $\bar{0}_T$, which equals $t_1 \otimes_T \bar{0}_T$.

In each case, then, there exists a t such that $t_{\mathrm{rhs}} = t_1 \otimes_T t$. Therefore $t_{\mathrm{rhs}} = t_1 \otimes_T t = t_1 \otimes_T t_{\mathrm{lhs}}$ follows from $\mathrm{LK}(T)$, and the theorem is proved.

If $\oplus_S$ is not selective, then the theorem must be about add_zero(T) rather than T, since otherwise $S \vec{\times} T$ is not defined. But $\mathrm{LD}(T) \implies \mathrm{LD}(\text{add\_zero}(T))$ and $\mathrm{LK}(T) \implies \mathrm{LK}(\text{add\_zero}(T))$, so only the T in the consequent would need to be replaced.

2 $\mathrm{NLD}(S) \vee \mathrm{NLD}(T) \vee (\mathrm{NLC}(S) \wedge \mathrm{NLK}(T)) \implies \mathrm{NLD}(S \vec{\times} T)$

Any disjunction is logically equivalent to one where earlier disjuncts are assumed not to hold in later ones. So the antecedent above is equivalent to $\mathrm{NLD}(S) \vee (\mathrm{LD}(S) \wedge \mathrm{NLD}(T)) \vee (\mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge \mathrm{NLC}(S) \wedge \mathrm{NLK}(T))$. Disjunction distributes over implication, so we have three cases to prove.

2.1 $\mathrm{NLD}(S)$

From $\mathrm{NLD}(S)$ we get witnesses a, b, and c such that $c \otimes_S (a \oplus_S b) \neq (c \otimes_S a) \oplus_S (c \otimes_S b)$. We can construct witnesses for $\mathrm{NLD}(S \vec{\times} T)$ as follows. Let $t \in T$. Then:

$$\begin{aligned}
&(c, t) \otimes ((a, t) \vec{\oplus} (b, t)) \\
&= (c, t) \otimes (a \oplus_S b, t') \\
&= (c \otimes_S (a \oplus_S b), t'')
\end{aligned}$$

for some t' and t''. Also:

$$((c,t) \otimes (a,t))\vec{\oplus}((c,t) \otimes (b,t))$$
$$= (c \otimes_S a, t \otimes_T t)\vec{\oplus}(c \otimes_S b, t \otimes_T t)$$
$$= ((c \otimes_S a) \oplus_S (c \otimes_S b), t''')$$

for some t'''. If these were equal, then their first components would be equal. So we would have $c \otimes_S (a \oplus_S b) = (c \otimes_S a) \oplus_S (c \otimes_S b)$. But we already know a, b, and c do not satisfy this property. Therefore (a,t), (b,t), and (c,t), for any t, serve as witnesses for $\mathrm{NLD}(S \vec{\times} T)$.

Note we assumed T is not empty, and this condition should be added to the hypotheses of the theorem.

2.2 $\mathrm{LD}(S) \wedge \mathrm{NLD}(T)$

From $\mathrm{NLD}(T)$ we get witnesses a, b, and c such that $c \otimes_T (a \oplus_T b) \neq (c \otimes_T a) \oplus_T (c \otimes_T b)$. We can construct witnesses for $\mathrm{NLD}(S \vec{\times} T)$ as follows. Let $s \in S$ be such that $s \oplus_S s = s$. Consider:

$$(s,c) \otimes ((s,a)\vec{\oplus}(s,b))$$
$$= (s,c) \otimes (s \oplus_S s, a \oplus_T b) \qquad (s = s \oplus_S s = s)$$
$$= (s \otimes_S s, c \otimes_T (a \oplus_T b))$$

From $\mathrm{LD}(S)$ we obtain $(s \otimes_S s) \oplus_S (s \otimes_S s) = s \otimes_S (s \oplus_S s) = s \otimes_S s$. Thus, we have:

$$((s,c) \otimes (s,a))\vec{\oplus}((s,c) \otimes (s,b))$$
$$= (s \otimes_S s, c \otimes_T a)\vec{\oplus}(s \otimes_S s, c \otimes_T b)$$
$$= ((s \otimes_S s) \oplus_S (s \otimes_S s), (c \otimes_T a) \oplus_T (c \otimes_T b))$$

If these were equal, then their second components would be equal. So we would have $c \otimes_T (a \oplus_T b) = (c \otimes_T a) \oplus_T (c \otimes_S b)$. But we already know a, b, and c do not satisfy this property. Therefore (s,a), (s,b), and (s,c), for any suitable s, serve as witnesses for $\mathrm{NLD}(S \vec{\times} T)$.

We assumed that there was an $s \in S$ such that $s \oplus_S s = s$, and this should be added to the hypotheses. This would be true if $\oplus_S$ were selective[1] or idempotent, or if $\bar{0}_S \in S$.

2.3 $\mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge \mathrm{NLC}(S) \wedge \mathrm{NLK}(T)$

From $\mathrm{NLC}(S)$ we get witnesses a, b, and c such that $c \otimes_S a = c \otimes_S b$ and $a \neq b$. From $\mathrm{NLK}(T)$ we get witnesses d, e, and f such that $f \otimes_T d \neq f \otimes_T e$. We proceed by cases on $a \oplus_S b$. There are only three cases because $a \neq b$.

1. $a = a \oplus_S b$
   We know $a \oplus_S b \neq b$ since $a \neq b$. Our witnesses for $\mathrm{NLD}(S \vec{\times} T)$ depend on $(f \otimes_T d) \oplus_T (f \otimes_T e)$. Since $f \otimes_T d \neq f \otimes_T e$, there are only three subcases.

   (a) $(f \otimes_T d) = (f \otimes_T d) \oplus_T (f \otimes_T e)$
   Take as witnesses (a,e), (b,d), and (c,f), and observe:

$$(c,f) \otimes ((a,e)\vec{\oplus}(b,d))$$
$$= (c,f) \otimes (a,e)$$
$$= (c \otimes_S a, f \otimes_T e)$$

$$((c,f) \otimes (a,e))\vec{\oplus}((c,f) \otimes (b,d))$$
$$= ((c \otimes_S a, f \otimes_T e)\vec{\oplus}(c \otimes_S b, f \otimes_T d)$$
$$= (c \otimes_S (a \oplus_S b), (f \otimes_T e) \oplus_T (f \otimes_T d))$$
$$= (c \otimes_S a, f \otimes_T d)$$

[1] selectivity implies idempotence

The second components are not equal, since $f \otimes_T e \neq f \otimes_T d$. Thus we have witnesses for $\mathrm{NLD}(S \vec{\times} T)$.

(b) $(f \otimes_T d) \oplus_T (f \otimes_T e) = f \otimes_T e$
Take as witnesses (a, d), (b, e), and (c, f), and observe:

$$\begin{array}{ll} (c,f) \otimes ((a,d) \vec{\oplus} (b,e)) & ((c,f) \otimes (a,d)) \vec{\oplus} ((c,f) \otimes (b,e)) \\ = (c,f) \otimes (a,d) & = ((c \otimes_S a, f \otimes_T d) \vec{\oplus} (c \otimes_S b, f \otimes_T e) \\ = (c \otimes_S a, f \otimes_T d) & = (c \otimes_S (a \oplus_S b), (f \otimes_T d) \oplus_T (f \otimes_T e)) \\ & = (c \otimes_S a, f \otimes_T e) \end{array}$$

The second components are not equal, since $f \otimes_T d \neq f \otimes_T e$. Thus we have witnesses for $\mathrm{NLD}(S \vec{\times} T)$.

(c) $f \otimes_T d \neq (f \otimes_T d) \oplus_T (f \otimes_T e) \neq f \otimes_T e$
Take the same witnesses as in the previous subcase. Since nothing in S has changed, both derivations stay the same, except for the final simplification on the right. But the second components are still not equal, since $f \otimes_S d \neq (f \otimes_T d) \oplus_T (f \otimes_T e)$ by assumption. (The witnesses from the first subcase would also work here.)

2. $a \oplus_S b = b$
A symmetrical argument to the one for the previous case works for this case. The subcases are based on $(f \otimes_T e) \oplus_T (f \otimes_T d)$ (being $f \otimes_T e$, $f \otimes_T d$, or neither), and the witness sets come in the same order as before.

3. $a \neq a \oplus_S b \neq b$
Considering the same two sets of witnesses as before, we have $(c, f) \otimes ((a, d) \vec{\oplus} (b, e)) = (c, f) \otimes ((a, e) \vec{\oplus} (b, d)) = (c \otimes_S (a \oplus_S b), \bar{0}_T)$ for the undistributed term.

Now this case, which only arises when $\oplus_S$ is not selective, only works if we make extra assumptions. The extra assumptions are:

- $c \otimes_S a = c \otimes_S (a \oplus_S b)$, which would follow if $\oplus_S$ were idempotent since $c \otimes_S a = c \otimes_S b$ and $\mathrm{LD}(S)$; and,
- either $f \otimes_T (d \oplus_T e) \neq \bar{0}_T$ or $f \otimes_T (e \oplus_T d) \neq \bar{0}_T$.

Given just the first extra assumption, the distributed term simplifies as follows.

$$\begin{array}{ll} ((c,f) \otimes (a,d)) \vec{\oplus} ((c,f) \otimes (b,e)) & ((c,f) \otimes (a,e)) \vec{\oplus} ((c,f) \otimes (b,d)) \\ = (c \otimes_S a, f \otimes_T d) \vec{\oplus} (c \otimes_S b, f \otimes_T e) & = (c \otimes_S a, f \otimes_T e) \vec{\oplus} (c \otimes_S b, f \otimes_T d) \\ = (c \otimes_S (a \oplus_S b), f \otimes_T (d \oplus_T e)) & = (c \otimes_S (a \oplus_S b), f \otimes_T (e \oplus_T d)) \end{array}$$

Then using the second assumption, we pick either the witnesses on the left or the witnesses on the right to make the second component of one the terms above unequal to the second component, $\bar{0}_T$, of the undistributed term.