L11 Problem Set 3

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1 Scoped product example

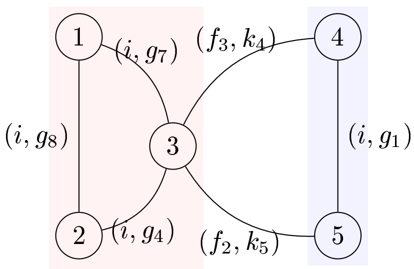
Let $S = (\mathbb{N}_\infty, \min, F \cup \{\omega, i\})$, where $i(x) = x$, $\omega(x) = \infty$, and $F = \{f_n(x) = x + n \mid n \in \mathbb{N}\}$. This is an algebra of monoid endomorphisms, since $f_n(\min(a, b)) = \min(a, b) + n = \min(a + n, b+n) = \min(f_n(a), f_n(b))$ and $f_n(\infty) = \infty+n = \infty$. Let $T = (\mathbb{N}_\infty, \max, G \cup \{\omega\})$, where $G = \{g_n(x) = nx \mid n \in \mathbb{N}\}$. This is also an algebra of monoid endomorphisms, since $g_n(\max(a, b)) = n \max(a, b) = \max(na, nb)$ and $g_n(\infty) = n\infty = \infty$.

The scoped product of these algebras is given by

$$S \Theta T = (\mathbb{N}_\infty \times \mathbb{N}_\infty, \min \vec{\times} \max, (F \times K) \cup (\{i\} \times G))$$

where $K = \{k_n(x) = n \mid n \in \mathbb{N}_\infty\}$.

Here is a graph over this algebra.



The adjacency matrix for this network is as follows.

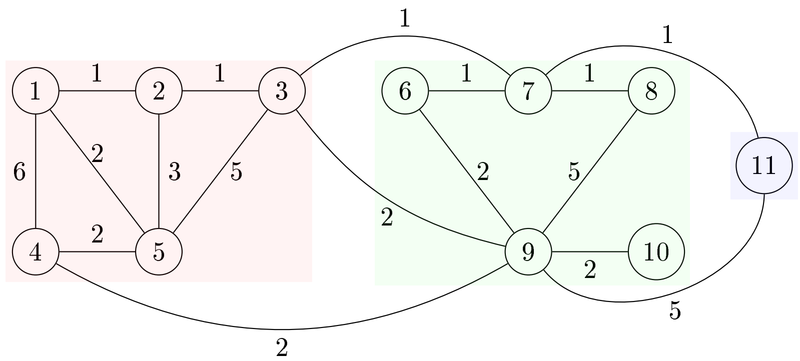
$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} (\omega, \omega) & (i, g_8) & (i, g_7) & (\omega, \omega) & (\omega, \omega) \\ (i, g_8) & (\omega, \omega) & (i, g_4) & (\omega, \omega) & (\omega, \omega) \\ (i, g_7) & (i, g_4) & (\omega, \omega) & (f_3, k_4) & (f_2, k_5) \\ (\omega, \omega) & (\omega, \omega) & (f_3, k_4) & (\omega, \omega) & (i, g_1) \\ (\omega, \omega) & (\omega, \omega) & (f_2, k_5) & (i, g_1) & (\omega, \omega) \end{array}\right] \end{array}$$

The routing matrix for this network is as follows.

$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} (i, i) & (i, g_8) & (i, g_7) & (f_2, k_{35}) & (f_2, k_5) \\ (i, g_8) & (i, i) & (i, g_7) & (f_2, k_{20}) & (f_2, k_{20}) \\ (i, g_7) & (i, g_7) & (i, i) & (f_2, k_5) & (f_2, k_5) \\ (f_2, k_{35}) & (f_2, k_{20}) & (f_2, k_5) & (i, i) & (i, g_1) \\ (f_2, k_5) & (f_2, k_{20}) & (f_2, k_5) & (i, g_1) & (i, i) \end{array}\right] \end{array}$$

2 Metric-neutral partitions example

Consider the following graph in the context of shortest-path routing, that is, in the matrix semiring over $(\mathbb{N}_\infty, \min, +, \infty, 0)$.



The corresponding adjacency matrix is

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{ccccccccccc}
\infty & 1 & \infty & 6 & 2 & \infty & \infty & \infty & \infty & \infty & \infty \\
1 & \infty & 1 & \infty & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & 1 & \infty & \infty & 5 & \infty & 1 & \infty & 2 & \infty & \infty \\
6 & \infty & \infty & \infty & 2 & \infty & \infty & \infty & 2 & \infty & \infty \\
2 & 3 & 5 & 2 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 2 & \infty & \infty \\
\infty & \infty & 1 & \infty & \infty & 1 & \infty & 1 & \infty & \infty & 1 \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 5 & \infty & \infty \\
\infty & \infty & 2 & 2 & \infty & 2 & \infty & 5 & \infty & 2 & 5 \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 2 & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 5 & \infty & \infty
\end{array}\right] \end{array}$$

We can calculate the routing matrix directly.

$$\mathbf{A}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{ccccccccccc}
0 & 1 & 2 & 4 & 2 & 4 & 3 & 4 & 4 & 6 & 4 \\
1 & 0 & 1 & 5 & 3 & 3 & 2 & 3 & 3 & 5 & 3 \\
2 & 1 & 0 & 4 & 4 & 2 & 1 & 2 & 2 & 4 & 2 \\
4 & 5 & 4 & 0 & 2 & 4 & 5 & 6 & 2 & 4 & 6 \\
2 & 3 & 4 & 2 & 0 & 6 & 5 & 6 & 4 & 6 & 6 \\
4 & 3 & 2 & 4 & 6 & 0 & 1 & 2 & 2 & 4 & 2 \\
3 & 2 & 1 & 5 & 5 & 1 & 0 & 1 & 3 & 5 & 1 \\
4 & 3 & 2 & 6 & 6 & 2 & 1 & 0 & 4 & 6 & 2 \\
4 & 3 & 2 & 2 & 4 & 2 & 3 & 4 & 0 & 2 & 4 \\
6 & 5 & 4 & 4 & 6 & 4 & 5 & 6 & 2 & 0 & 6 \\
4 & 3 & 2 & 6 & 6 & 2 & 1 & 2 & 4 & 6 & 0
\end{array}\right] \end{array}$$

However, it is worthwhile investigating how the routing matrix would be calculated using metric neutral partitions. There are three regions: nodes 1–5, nodes 6–10, and node 11. The adjacency matrix is decomposed as follows.

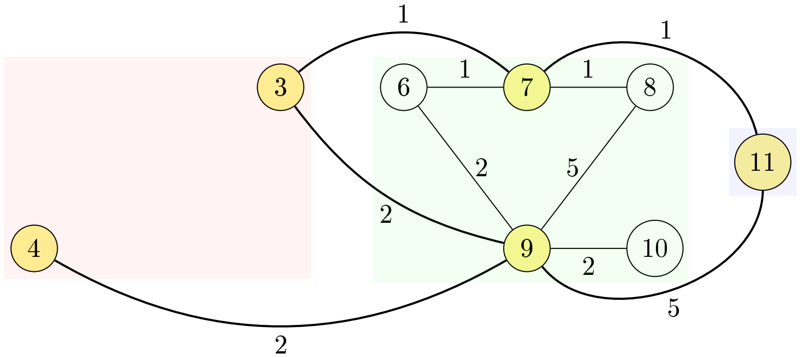
$$\mathbf{R} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1&2&3&4&5&6&7&8&9&10&11 \end{array} \\ \left[\begin{array}{ccccccccccc}
\infty & 1 & \infty & 6 & 2 & \infty & \infty & \infty & \infty & \infty & \infty \\
1 & \infty & 1 & \infty & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & 1 & \infty & \infty & 5 & \infty & \infty & \infty & \infty & \infty & \infty \\
6 & \infty & \infty & \infty & 2 & \infty & \infty & \infty & \infty & \infty & \infty \\
2 & 3 & 5 & 2 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 2 & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & 1 & \infty & 1 & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 5 & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & 2 & \infty & 5 & \infty & 2 & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 2 & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty
\end{array}\right] \end{array}$$

$$\mathbf{B} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1&2&3&4&5&6&7&8&9&10&11 \end{array} \\ \left[\begin{array}{ccccccccccc}
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 2 & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 2 & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & 1 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & 2 & 2 & \infty & \infty & \infty & \infty & \infty & \infty & 5 \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & \infty & 5 & \infty & \infty
\end{array}\right] \end{array}$$

The first interesting thing about this network can be seen if we abbreviate $\mathbf{B}$ by not printing rows and columns without any finite weights, and then reorder the nodes.

$$\mathbf{B} = \begin{array}{c} \\ 3\\4\\7\\9\\11 \end{array} \begin{array}{c} \begin{array}{ccccc} 3&4&7&9&11 \end{array} \\ \left[\begin{array}{ccccc}
\infty & \infty & 1 & 2 & \infty \\
\infty & \infty & \infty & 2 & \infty \\
1 & \infty & \infty & \infty & 1 \\
2 & 2 & \infty & \infty & 5 \\
\infty & \infty & 1 & 5 & \infty
\end{array}\right] \end{array} \qquad \hat{\mathbf{B}} = \begin{array}{c} \\ 3\\7\\11\\9\\4 \end{array} \begin{array}{c} \begin{array}{ccccc} 3&7&11&9&4 \end{array} \\ \left[\begin{array}{ccccc}
\infty & 1 & \infty & 2 & \infty \\
1 & \infty & 1 & \infty & \infty \\
\infty & 1 & \infty & 5 & \infty \\
2 & \infty & 5 & \infty & 2 \\
\infty & \infty & \infty & 2 & \infty
\end{array}\right] \end{array}$$

Now look at $\mathbf{R}$. We see that $\hat{\mathbf{B}} = \mathbf{A}_{2,2}$. This relationship is emphasised in the graph below, showing only the boundary subgraph and region 2.



The next step is to solve the region routing problem.

$\mathbf{R}^* =$

	1	2	3	4	5	6	7	8	9	10	11
1	0	1	2	4	2	∞	∞	∞	∞	∞	∞
2	1	0	1	5	3	∞	∞	∞	∞	∞	∞
3	2	1	0	6	4	∞	∞	∞	∞	∞	∞
4	4	5	6	0	2	∞	∞	∞	∞	∞	∞
5	2	3	4	2	0	∞	∞	∞	∞	∞	∞
6	∞	∞	∞	∞	∞	0	1	2	2	4	∞
7	∞	∞	∞	∞	∞	1	0	1	3	5	∞
8	∞	∞	∞	∞	∞	2	1	0	4	6	∞
9	∞	∞	∞	∞	∞	2	3	4	0	2	∞
10	∞	∞	∞	∞	∞	4	5	6	2	0	∞
11	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	0

From this we extract the transit matrix.

$\mathbf{T} =$

	3	4	7	9	11
3	0	6	∞	∞	∞
4	6	0	∞	∞	∞
7	∞	∞	0	3	∞
9	∞	∞	3	0	∞
11	∞	∞	∞	∞	0

Then we compute $\mathbf{C} = \mathbf{B} \oplus \mathbf{T}$. The second interesting thing about this network can be seen by reordering nodes in the core graph.

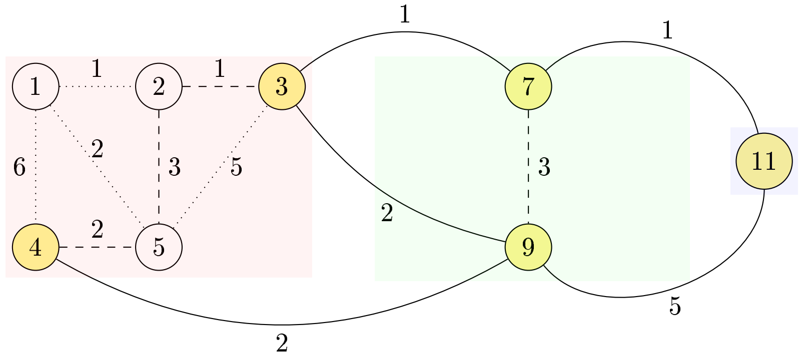
$\mathbf{C} =$

	3	4	7	9	11
3	0	6	1	2	∞
4	6	0	∞	2	∞
7	1	∞	0	3	1
9	2	2	3	0	5
11	∞	∞	1	5	0

$\hat{\mathbf{C}} =$

	3	7	11	4	9
3	0	1	∞	6	2
7	1	0	1	∞	3
11	∞	1	0	∞	5
4	6	∞	∞	0	2
9	2	3	5	2	0

Look again at $\mathbf{R}^*$. We see that $\hat{\mathbf{C}} = \mathbf{A}_{1,1}$. This relationship is emphasised in the following picture of the core graph plus internal edges for region 1. The edges in region 1 are shown dotted, except for the transit edge from 4 to 3 of weight 6. In fact there is also an alternative transit edge of the same weight, not emphasised in the picture.



Solving the region routing problem $\mathbf{R}^*$ involves computing $\mathbf{A}^*_{1,1}$. But since $\hat{\mathbf{C}} = \mathbf{A}_{1,1}$, solving the core routing problem $\mathbf{C}^*$ involves computing the same solution. The metric neutral partitions method would compute the same thing twice, possibly using different algorithms, in this unusual network. The solution for the core graph is given below; compare it to $\mathbf{R}^*$.

$$\mathbf{C}^* = \begin{array}{c} \scriptstyle 3 \\ \scriptstyle 4 \\ \scriptstyle 7 \\ \scriptstyle 9 \\ \scriptstyle 11 \end{array} \overset{\begin{array}{ccccc} \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 7 & \scriptstyle 9 & \scriptstyle 11 \end{array}}{\begin{bmatrix} 0 & 4 & 1 & 2 & 2 \\ 4 & 0 & 5 & 2 & 6 \\ 1 & 5 & 0 & 3 & 1 \\ 2 & 2 & 3 & 0 & 4 \\ 2 & 6 & 1 & 4 & 0 \end{bmatrix}} \qquad \hat{\mathbf{C}}^* = \begin{array}{c} \scriptstyle 3 \\ \scriptstyle 7 \\ \scriptstyle 11 \\ \scriptstyle 4 \\ \scriptstyle 9 \end{array} \overset{\begin{array}{ccccc} \scriptstyle 3 & \scriptstyle 7 & \scriptstyle 11 & \scriptstyle 4 & \scriptstyle 9 \end{array}}{\begin{bmatrix} 0 & 1 & 2 & 4 & 2 \\ 1 & 0 & 1 & 5 & 3 \\ 2 & 1 & 0 & 6 & 4 \\ 4 & 5 & 6 & 0 & 2 \\ 2 & 3 & 4 & 2 & 0 \end{bmatrix}}$$

Of course, the routing matrix $\mathbf{A}^*$ given at the start of the section can also be computed as $\mathbf{R}^* \oplus \mathbf{R}^*\mathbf{C}^*\mathbf{R}^*$.

3 $(\mathbf{X} \oplus \mathbf{Y})^* = \mathbf{X}^*(\mathbf{Y}\mathbf{X}^*)^*$

Define a *simple* term as a product of constants or variables ranging over semiring elements. For example $\mathbf{X}^k$ and $\mathbf{Y}^a\mathbf{X}^b$ are simple, but $(\mathbf{X} \oplus \mathbf{Y})^k$ and $\mathbf{Y}^*$ are not. Any expression built using constants, variables, the $\oplus$ operator, and the $\otimes$ operator, can be inductively shown to have the same value as a sum of simple terms. Since the closure operator $(\cdot)^*$ is defined in terms of $\oplus$ and $\otimes$, we can say the same about expressions that also include it. If we assume $\oplus$ is commutative, we only need consider the multiset of simple terms in such a sum: the order does not matter, but the multiplicity might (since $\oplus$ might not be idempotent).

We define by recursion an operator $[\cdot]$ that denotes a multiset of simple terms, and prove by induction that the sum of terms in $[e]$ is equal to the value of e. We use set notation for the analogous multiset operations.

- For a variable, $[\mathbf{X}] = \{\mathbf{X}\}$; for a constant $[\mathbf{A}] = \{\mathbf{A}\}$.
- For a sum $[e_1 \oplus e_2] = [e_1] \cup [e_2]$.
- For a product $[e_1 \otimes e_2] = \{s_1 \otimes s_2 \mid s_1 \in [e_1], s_2 \in [e_2]\}$.

Note that the above definitions give us $[e^*] = [e^0 \oplus e^1 \oplus e^2 \oplus e^3 \oplus \dots] = [\mathbf{I} \oplus e \oplus (e \otimes e) \oplus (e \otimes e \otimes e) \oplus \dots] = [\mathbf{I}] \cup [e] \cup [e \otimes e] \cup [e \otimes e \otimes e] \cup \cdots = \{\mathbf{I}\} \cup \{s_1 \mid s_1 \in [e]\} \cup \{s_1 \otimes s_2 \mid s_1, s_2 \in [e]\} \cup \{s_1 \otimes s_2 \otimes s_3 \mid s_1, s_2, s_3 \in [e]\} \cup \cdots = \{s_1 \otimes s_2 \otimes \cdots \otimes s_k \mid k \in \mathbb{N}, \forall i \leq k.\ s_i \in [e]\}$.

Now the proof. The notation $\bigoplus M$, where M is a multiset, is short for $\bigoplus_{t \in M} t$.

- Clearly $\mathbf{X}$ is equal to $\bigoplus[\mathbf{X}] = \bigoplus\{\mathbf{X}\} = \mathbf{X}$, and similarly for constants.
- By induction, assume $e_1 = \bigoplus[e_1]$ and $e_2 = \bigoplus[e_2]$. Then $e_1 \oplus e_2 = \bigoplus[e_1] \oplus \bigoplus[e_2] = \bigoplus([e_1] \cup [e_2]) = \bigoplus[e_1 \oplus e_2]$.
- By induction, assume $e_1 = \bigoplus[e_1] = s_1 \oplus s_2 \oplus \cdots \oplus s_j$ and $e_2 = \bigoplus[e_2] = t_1 \oplus t_2 \oplus \cdots \oplus t_k$. Then $e_1 \otimes e_2 = \bigoplus[e_1] \otimes \bigoplus[e_2] = (s_1 \oplus s_2 \oplus \cdots \oplus s_j)(t_1 \oplus t_2 \oplus \cdots \oplus t_k) = s_1t_1 \oplus s_1t_2 \oplus \ldots s_1t_k \oplus s_2t_1 \oplus s_2t_2 \cdots \oplus \cdots \oplus s_jt_k = \bigoplus\{st \mid s \in [e_1], t \in [e_2]\} = \bigoplus[e_1 \otimes e_2]$. Note that we made use of distributivity here.

Thus we have $e = \bigoplus[e]$ for all expressions e (built as described above).

Consider the sum of simple terms equivalent to the left hand side of our equation.

$$\begin{aligned} [(\mathbf{X} \oplus \mathbf{Y})^*] &= \left\{ \bigotimes_{i=0}^{k} s_i \mid k \in \mathbb{N}, \forall i.\ s_i \in [\mathbf{X} \oplus \mathbf{Y}] \right\} \\ &= \left\{ \bigotimes_{i=0}^{k} s_i \mid k \in \mathbb{N}, \forall i.\ s_i \in \{\mathbf{X}, \mathbf{Y}\} \right\} \end{aligned}$$

Thus $(\mathbf{X} \oplus \mathbf{Y})^*$ is equal to the sum of all products of $\mathbf{X}$s and $\mathbf{Y}$s.

Now consider the sum of simple terms equivalent to the right hand side, which we build up in pieces.

$$\begin{aligned} [\mathbf{X}^*] &= \left\{ \bigotimes_{i=0}^{k} s_i \mid k \in \mathbb{N}, \forall i.\ s_i \in [\mathbf{X}] \right\} \\ &= \left\{ \bigotimes_{i=0}^{k} \mathbf{X} \mid k \in \mathbb{N} \right\} \\ &= \left\{ \mathbf{X}^k \mid k \in \mathbb{N} \right\} \\ [\mathbf{Y}\mathbf{X}^*] &= \{s_1 \otimes s_2 \mid s_1 \in [\mathbf{Y}], s_2 \in [\mathbf{X}^*]\} \\ &= \{\mathbf{Y}\mathbf{X}^k \mid k \in \mathbb{N}\} \\ [(\mathbf{Y}\mathbf{X}^*)^*] &= \left\{ \bigotimes_{i=0}^{k} s_i \mid k \in \mathbb{N}, \forall i.\ s_i \in [\mathbf{Y}\mathbf{X}^*] \right\} \\ &= \left\{ \bigotimes_{i=0}^{k} \mathbf{Y}\mathbf{X}^{j_i} \mid k \in \mathbb{N}, \forall i.\ j_i \in \mathbb{N} \right\} \\ [\mathbf{X}^*(\mathbf{Y}\mathbf{X}^*)^*] &= \{s_1 \otimes s_2 \mid s_1 \in [\mathbf{X}^*], s_2 \in [(\mathbf{Y}\mathbf{X}^*)^*]\} \\ &= \left\{ \mathbf{X}^k \otimes \bigotimes_{i=0}^{j} \mathbf{Y}\mathbf{X}^{l_i} \mid k, j \in \mathbb{N}, \forall i.\ l_i \in \mathbb{N} \right\} \end{aligned}$$

Finally, we show that $[(\mathbf{X} \oplus \mathbf{Y})^*] = [\mathbf{X}^*(\mathbf{Y}\mathbf{X}^*)^*]$. Our notation has already betrayed our assumption that $\otimes$ is associative. Given associativity, we can think of a product $\bigotimes_{i=0}^{k} s_i$, where each $s_i \in \{\mathbf{X}, \mathbf{Y}\}$, as a sequence of $\mathbf{X}$s and $\mathbf{Y}$s of length k. To each such sequence we can assign a unique sequence of the form $\mathbf{X}^h \otimes \bigotimes_{i=0}^{j} \mathbf{Y}\mathbf{X}^{l_i}$, and vice versa, and thereby show that the two multisets are equal.

- ($\subseteq$) Let $s = \bigotimes_{i=0}^{k} s_i \in [(\mathbf{X} \oplus \mathbf{Y})^*]$. Take j as the number of $\mathbf{Y}$s in the sequence s. After each $\mathbf{Y}$ except the last, there is a sequence of $\mathbf{X}$s before the next $\mathbf{Y}$. Take l_i as the length of the sequence of $\mathbf{X}$s after the $(i+1)^{\text{th}}$ $\mathbf{Y}$. Finally, take h as the length of the sequence of $\mathbf{X}$s before the first $\mathbf{Y}$. Then $\mathbf{X}^h \otimes \bigotimes_{i=0}^{j} \mathbf{Y}\mathbf{X}^{l_i}$ is the same sequence as s by construction.

  For any other choices of h, j, and l_i, the sequence will obviously differ from s, because our choices were completely determined by s. Therefore the corresponding sequence is unique.

- ($\supseteq$) Obviously any $\mathbf{X}^h \otimes \bigotimes_{i=0}^{j} \mathbf{Y}\mathbf{X}^{l_i}$ is a sequence of $\mathbf{X}$s and $\mathbf{Y}$s. It is a particular sequence, and therefore unique in $[(\mathbf{X} \oplus \mathbf{Y})^*]$.

  Note that the only way for elements of $[(\mathbf{X} \oplus \mathbf{Y})^*]$ to occur with multiplicity greater than 1 is if $\mathbf{X} = \mathbf{Y}$. But if we formally treat the two variables in $\mathbf{X} \oplus \mathbf{Y}$ as different (and similarly in $\mathbf{X}^*(\mathbf{Y}\mathbf{X}^*)^*$), the bijection between our multisets is more easily seen. A substitution of $\mathbf{X}$ for $\mathbf{Y}$ can be done at the end.

We may now conclude our desired equality, reasoning as follows:

$$(\mathbf{X} \oplus \mathbf{Y})^* = \bigoplus[(\mathbf{X} \oplus \mathbf{Y})^*] = \bigoplus[\mathbf{X}^*(\mathbf{Y}\mathbf{X}^*)^*] = \mathbf{X}^*(\mathbf{Y}\mathbf{X}^*)^*$$

4 $\mathbf{A}^* = \mathbf{R}^* \oplus \mathbf{R}^*\mathbf{C}^*\mathbf{R}^*$

We assume the underlying semiring is 0-stable. We will make use of the following lemmas, which will be proved after the main theorem.

1. for all $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X}(\mathbf{Y}^*\mathbf{X})^* = (\mathbf{X}\mathbf{Y}^*)^*\mathbf{X}$
2. for all $\mathbf{X}$, $\mathbf{X}^* = \mathbf{I} \oplus \mathbf{X}^*\mathbf{X}$
3. $\mathbf{T}^* = \mathbf{I} \oplus \mathbf{T}$
4. $\mathbf{B}\mathbf{R}^*\mathbf{B} = \mathbf{B}\mathbf{T}^*\mathbf{B}$
5. $\mathbf{R}^*\mathbf{T}^* = \mathbf{R}^* = \mathbf{T}^*\mathbf{R}^*$
6. $\mathbf{R}^* = \mathbf{R}^* \oplus \mathbf{R}^*\mathbf{R}^*$

Now we reason as follows.

$$\begin{aligned}
\mathbf{A}^* &= (\mathbf{R}+\mathbf{B})^* \\
&= \mathbf{R}^*(\mathbf{B}\mathbf{R}^*)^* && \text{by previous section} \\
&= \mathbf{R}^*(\mathbf{I} \oplus \mathbf{B}\mathbf{R}^* \oplus \mathbf{B}\mathbf{R}^*\mathbf{B}\mathbf{R}^* \oplus \mathbf{B}\mathbf{R}^*\mathbf{B}\mathbf{R}^*\mathbf{B}\mathbf{R}^* \dots) \\
&= \mathbf{R}^*(\mathbf{I} \oplus \mathbf{B}(\mathbf{R}^*\mathbf{B})^*\mathbf{R}^*) \\
&= \mathbf{R}^*(\mathbf{I} \oplus (\mathbf{B} \oplus \mathbf{B}\mathbf{R}^*\mathbf{B} \oplus \mathbf{B}\mathbf{R}^*\mathbf{B}\mathbf{R}^*\mathbf{B} \oplus \dots)\mathbf{R}^*) \\
&= \mathbf{R}^*(\mathbf{I} \oplus (\mathbf{B} \oplus \mathbf{B}\mathbf{T}^*\mathbf{B} \oplus \mathbf{B}\mathbf{T}^*\mathbf{B}\mathbf{T}^*\mathbf{B} \oplus \dots)\mathbf{R}^*) && \text{by lemma 4} \\
&= \mathbf{R}^*(\mathbf{I} \oplus \mathbf{B}(\mathbf{T}^*\mathbf{B})^*\mathbf{R}^*) \\
&= \mathbf{R}^*(\mathbf{I} \oplus (\mathbf{B}\mathbf{T}^*)^*\mathbf{B}\mathbf{T}^*\mathbf{R}^*) && \text{by lemma 1} \\
&= \mathbf{R}^* \oplus \mathbf{R}^*(\mathbf{B}\mathbf{T}^*)^*\mathbf{B}\mathbf{T}^*\mathbf{R}^* \\
&= \mathbf{R}^* \oplus \mathbf{R}^*\mathbf{R}^* \oplus \mathbf{R}^*(\mathbf{B}\mathbf{T}^*)^*\mathbf{B}\mathbf{T}^*\mathbf{R}^* && \text{by lemma 6} \\
&= \mathbf{R}^* \oplus \mathbf{R}^*(\mathbf{I} \oplus (\mathbf{B}\mathbf{T}^*)^*\mathbf{B}\mathbf{T}^*)\mathbf{R}^* \\
&= \mathbf{R}^* \oplus \mathbf{R}^*(\mathbf{B}\mathbf{T}^*)^*\mathbf{R}^* && \text{by lemma 2} \\
&= \mathbf{R}^* \oplus \mathbf{R}^*\mathbf{T}^*(\mathbf{B}\mathbf{T}^*)^*\mathbf{R}^* && \text{by lemma 5} \\
&= \mathbf{R}^* \oplus \mathbf{R}^*(\mathbf{T} \oplus \mathbf{B})^*\mathbf{R}^* && \text{by previous section} \\
&= \mathbf{R}^* \oplus \mathbf{R}^*\mathbf{C}^*\mathbf{R}^*
\end{aligned}$$

We turn now to the lemmas. First, lemma 1.

$$\begin{aligned}
\mathbf{X}(\mathbf{Y}^*\mathbf{X})^* &= \mathbf{X} \oplus \mathbf{X}\mathbf{Y}^*\mathbf{X} \oplus \mathbf{X}\mathbf{Y}^*\mathbf{X}\mathbf{Y}^*\mathbf{X} \oplus \dots \\
&= (\mathbf{X}\mathbf{Y}^*)\mathbf{X}
\end{aligned}$$

Now, lemma 2.

$$\begin{aligned}
\mathbf{X}^* &= \mathbf{I} \oplus \mathbf{X} \oplus \mathbf{X}^2 \oplus \mathbf{X}^3 \oplus \dots \\
&= \mathbf{I} \oplus (\mathbf{I} \oplus \mathbf{X} \oplus \mathbf{X}^2 \oplus \dots)\mathbf{X} \\
&= \mathbf{I} \oplus \mathbf{X}^*\mathbf{X}
\end{aligned}$$

Now lemma 3. It suffices to show $\mathbf{T}^(1) = \mathbf{T}^(2)$, since then $\mathbf{T}^* = \mathbf{T}^(1) = \mathbf{I} \oplus \mathbf{T}$. We prove the result for each entry (i, j). If $i = j$, then we have

$$\begin{aligned}
\mathbf{T}^(2)(i,i) &= \mathbf{I}(i,i) \oplus \mathbf{T}(i,j) \oplus \mathbf{T}^2(i,j) \\
&= \bar{1} \oplus \dots \\
&= \bar{1} && \text{by 0-stability} \\
&= \mathbf{I}(i,i) \oplus \mathbf{T}(i,i) \\
&= \mathbf{T}^(1)
\end{aligned}$$

If $i \neq j$ but (i, j) is not a transit arc, then either i or j is not a border node. So at least one arc out of (i, q) and (q, j) must not be a transit arc, for any q. We reason as follows.

$$
\begin{aligned}
\mathbf{T}^(2)(i,j) &= \mathbf{I}(i,j) \oplus \mathbf{T}(i,j) \oplus \mathbf{T}^2(i,j) \\
&= \bar{0} \oplus \mathbf{T}(i,j) \oplus \bigoplus_q \mathbf{T}(i,q)\mathbf{T}(q,j) \\
&= \bar{0} \oplus \bar{0} \oplus \bigoplus_q \mathbf{T}(i,q)\mathbf{T}(q,j) && \mathbf{T} \text{ is } \bar{0} \text{ on non-transit arcs} \\
&= \bar{0} \oplus \bar{0} \oplus \bigoplus_q \bar{0}\mathbf{T}(q,j) && (\text{or } T(i,q)\bar{0}) \\
&= \bar{0} \\
&= \mathbf{T}(i,j)
\end{aligned}
$$

If $i \neq j$, and (i, j) is a transit arc, then $\mathbf{T}(i, j) = \mathbf{A}^*_{m,m}(i, j)$ for some region m. Furthermore, the only non-zero values for $\mathbf{T}(i, q)$ and $\mathbf{T}(q, j)$ will occur when q is also a border node in the same region m.

$$
\begin{aligned}
\mathbf{T}^(2)(i,j) &= \mathbf{I}(i,j) \oplus \mathbf{T}(i,j) \oplus \mathbf{T}^2(i,j) \\
&= \bar{0} \oplus \mathbf{T}(i,j) \oplus \bigoplus_q \mathbf{T}(i,q)\mathbf{T}(q,j) \\
&= \mathbf{A}^*_{m,m}(i,j) \oplus \bigoplus_q \mathbf{A}^*_{m,m}(i,q)\mathbf{A}^*_{m,m}(q,j) \\
&= \mathbf{A}^*_{m,m}(i,j) && \text{since } \oplus \text{ is idempotent} \\
&= \mathbf{T}(i,j)
\end{aligned}
$$

To explain the penultimate line, remember that in a 0-stable semiring, the addition operator is idempotent. Now $\mathbf{A}^*_{m,m}(i, j)$ stands for the sum over all paths from i to j in region m. All paths that pass through an intermediate vertex q are also included. Therefore all terms in the sum on the right are already in the sum on the left, and idempotency means only one copy of each term is required.

Now lemma 4. First we note that if x and y are nodes in the same region, then $\mathbf{B}(x, y) = \bar{0}$ by definition. Similarly if x and y are nodes in different regions, then $\mathbf{R}(x, y) = \bar{0}$, and it follows that $\mathbf{R}^*(x, y) = \bar{0}$. Consider the expansion of the left hand side:

$$
\mathbf{BR^*B}(i,j) = \bigoplus_{q_1} \bigoplus_{q_2} \mathbf{B}(i,q_1)\mathbf{R}^*(q_1,q_2)\mathbf{B}(q_2,j)
$$

We may restrict attention to q_1 and q_2 such that $r(i) \neq r(q_1)$ and $r(q_1) = r(q_2)$ and $r(q_2) \neq r(j)$, where $r(x)$ is the region of node x, since otherwise the term in the sum becomes $\bar{0}$. Now if $q_1 = q_2$ then $\mathbf{R}^*(q_1, q_2) = \bar{1}$ since the semiring is 0-stable. On the other hand, if $q_1 \neq q_2$ then we may assume (q_1, q_2) is a transit arc connecting $r(i)$ and $r(j)$, since otherwise either $\mathbf{B}(i, q_1)$

or $\mathbf{B}(q_2, j)$ will be $\bar{0}$. Therefore $\mathbf{R}^*(q_1, q_2) = \mathbf{T}(q_1, q_2)$ by definition of $\mathbf{T}$. Thus we have:

$$\begin{aligned}
\mathbf{B}\mathbf{R}^*\mathbf{B}(i,j) &= \bigoplus_{q_1}\bigoplus_{q_2} \mathbf{B}(i,q_1)\mathbf{R}^*(q_1,q_2)\mathbf{B}(q_2,j) \\
&= \bigoplus_{q} \mathbf{B}(i,q)\bar{1}\mathbf{B}(q,j) \oplus \bigoplus_{q_1 \neq q_2} \mathbf{B}(i,q_1)\mathbf{T}(q_1,q_2)\mathbf{B}(q_2,j) \\
&= \bigoplus_{q_1}\bigoplus_{q_2} \mathbf{B}(i,q_1)(\mathbf{I} \oplus \mathbf{T})(q_1,q_2)\mathbf{B}(q_2,j) \\
&= \bigoplus_{q_1}\bigoplus_{q_2} \mathbf{B}(i,q_1)\mathbf{T}^*(q_1,q_2)\mathbf{B}(q_2,j) && \text{by lemma 5} \\
&= \mathbf{B}\mathbf{T}^*\mathbf{B}(i,j)
\end{aligned}$$